Chapter 1

Introduction

Topics :

1. Motivation and Basic Concepts
2. Mathematical Formulation of the Control Problem
3. Examples
4. Matrix Theory (review)

A *control system* can be viewed informally as a dynamical object (e.g. ordinary differential equation) containing a parameter (control) which can be manipulated to influence the behaviour of the system so as to achieve a desired goal. In order to implement this influence, engineers build devices that incorporate various mathematical techniques. *Mathematical control theory* is today a well-established branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems.

1.1 Motivation and Basic Concepts

Mathematical control theory is a rapidly growing field which provides theoretical and computational tools for dealing with a variety of problems arising in *electrical* and *aerospace engineering, automatics, robotics, management, economics, applied chemistry, biology, ecology, medicine,* etc. Selected such problems, to mention but a few, are the following : stable performance of motors and machinery, optimal guidance of rockets, optimal exploitation of natural resources, optimal investment or production strategies, regulation of physiological functions, and fight against insects, epidemics.

All these (and many other) problems require a specific approach, the aim being *to compel or control a system to behave in some desired fashion.*

Systems

A **system** is something having parts which is perceived as a single entity.

NOTE : Not everything is a system (for instance, a *point* or the empty set). However, most things can usefully be seen as systems of some kind. A system is, so to speak, a world.

The parts making up a system may be clearly or vaguely defined. The interesting thing about a system is the way the parts are related to each other. For the systems studied in mathematics, the parts and their relations must be so clearly defined that we can single out a particular set of relations as completely characterizing the *state* of the system; then *we identify the system with the collection of all its conceivable states.* It seems to be necessary that the state space be clearly and unambiguously defined. Unfortunately this usually means that the mathematical system is drastically oversimplified in comparison with the *natural system* being modelled.

When attempting to study the behaviour of certain systems, it is convenient to consider the ideal case of an *"isolated system"* – i.e. a number of interacting elements which do not have any interaction with the rest of the

world. In reality, no system is ever completely isolated, but in many cases the interactions with the rest of the world can be neglected in a reasonable approximation. In such "isolated systems" the conditions are simpler, and therefore easier to study.

NOTE : Our Universe is, by definition, an isolated system.

Dynamical and control systems

A **dynamical system** is one which changes in time (in some well defined way); what changes is the state of the system. For such systems, the basic problem is to *predict* the future behaviour. For this purpose the differential equations are exactly tailored. The differential equation itself represents the (physical or otherwise) law governing the evolution of the system; this plus the initial conditions should determine uniquely the future evolution of the system.

NOTE : Philosophically this leads to *determinism*, and is independent of any (human) observer. Modern physics changed this view, to some extent, by making the observer a much more active participant in the possible outcome of future measurements. But even within classical physics, the prediction of future evolution is not the only meaningful problem to be posed. The whole field of engineering and technology deals, to a large extent, with the"inverse" problem : *given a desired future evolution, how should we construct the system?*

One can introduce some way of acting upon a (dynamical) system and influence its evolution (behaviour). We think of this outside action, also called *input* (or *control*), as the result of decisions of a "controller" (possibly human), who may have some definite goal in mind or not. But this last is irrelevant and the important information we need is a rule, within the description of the system, of which inputs are possible and which are not; the possible inputs will then be called "admissible". A simple example of such systems (with inputs) is a *car*, whose motion depends on the input of all the actions by which we drive it. In most cases, it is possible to change the state of the system in any

prescribed fashion by properly choosing the inputs, at least within reasonable limits. In other words, *one may exert influence on the system state by means of intelligent manipulation of its inputs.* This then, in a general sense, constitutes a **control system**.

The control engineer develops the techniques and hardware necessary for the implementation of the control laws to the specific systems in question.

NOTE : The complexity of many systems in present-day world is such that it is often desirable for control to be carried out *automatically*, without direct human intervention. To take a simple example, the room thermostat in a domestic central heating system turns the boiler on and off so as to maintain room temperature at a predetermined level. Nature provides many examples of remarkable self-regulation, such as the way in which body temperature is kept constant despite large variations in external conditions.

Some basic control-theoretic concepts

We summarize some of the main features of a control system.

The **state variables** $x_1, x_2, \dots, x_m$ describe the condition (or state) of the system, and provide the information which (together with the knowledge of the equations describing the system) enables us to calculate the future behaviour from the knowledge of the **control variables** (or **inputs**) $u_1, u_2, \dots, u_\ell$. In practice, it is often not possible to determine the values of the state variables directly; instead, only a set of **controlled variables** (or **outputs**) $y_1, y_2, \dots, y_n$, which depend in some way on the state variables, is measured. In general, *the aim is to make a system perform in some required way by suitably manipulating the inputs*, this being done by some controlling device (or "controller").

If the controller operates according to some pre-set pattern without taking account of the output or state, the system is called **open loop**. If, however, there is *feedback* of information concerning the outputs to controller, which then appropriately modifies its course of action, the system is called **closed loop**.

We assume that our system models have the property that, *given an initial state and any input, the resulting state and output at some specified later time are uniquely determined*.

1.2 Mathematical Formulation of the Control Problem

Roughly speaking, a *control system* is a dynamical system together with a class of "admissible inputs". We whish to make this idea more precise, without striving for full generality.

Control systems

We assume that the *dynamics* of the system, that is, the evolution of the **state vector** $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} \in \mathbb{R}^{m \times 1} = \mathbb{R}^m$ under a given **input** (or **control**) **vector** $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_\ell(t) \end{bmatrix} \in \mathbb{R}^{\ell \times 1} = \mathbb{R}^\ell$ is determined by a (vector) *ordinary differential equation*

$$\dot{x} = F(t, x, u). \tag{1.1}$$

The *input* vector $u(\cdot)$ is assumed to be an "arbitrary" vector-valued mapping, but some restrictions must be imposed. First of all, its components – the input functions $u_1(\cdot), u_2(\cdot), \ldots, u_\ell(\cdot)$ – must be *measurable* (think of *piecewise-continuous* functions), since otherwise the differential equation (1.1) wouldn't make sense.

NOTE : To restrict the control to be a *continuous* mapping, would be too much, since in many cases the piecewise-continuous inputs (with some points of discontinuity) are the most interesting controls.

Another restriction of the input vector is the requirement that the values of $u(\cdot)$ belong to a specified set U. (For example, when turning a steering wheel of a car, we are restricted to a maximum turning angle to either side.) Such a restriction is then of the form

$$u(t) \in U \subseteq \mathbb{R}^\ell. \tag{1.2}$$

An *admissible input* is therefore a piecewise-continuous (vector-valued) mapping $u(\cdot)$ satisfying (1.2). We denote by $\mathcal{U}$ the set of all these admissible inputs.

Furthermore, it is assumed that the vector-valued mapping $F : J \times \mathbb{R}^m \times U \to \mathbb{R}^m$, $J \subseteq \mathbb{R}$ satisfies certain standard conditions (such as having continuous first order partial derivatives).

NOTE : This assumption guarantees local existence and uniqueness of the solution of (1.1) (subject to *initial condition* $x(t_0) = x_0$) for a given $u(\cdot) \in \mathcal{U}$.

A **control system** is a 4-tuple

$$\Sigma = (M, U, \mathcal{U}, F).$$

In this case, the set $M = \mathbb{R}^m$ is the **state space**, the set $U \subseteq \mathbb{R}^\ell$ is the **control set**, $\mathcal{U}$ is the **input class**, and the mapping F is the **dynamics** of Σ. We say that the control system Σ is defined (or described) by the *state equation* (1.1) and write (in classical notation) :

$$\boxed{\Sigma : \quad \dot{x} = F(t, x, u), \qquad x \in M, \ \ u \in U \subseteq \mathbb{R}^\ell.}$$

NOTE : (1) In fact, such a system is a *continuous-time, time-varying, finite dimensional, differentiable (nonlinear) control system.*

(2) The state space M carries certain (geometric) "structure". It is natural to assume that M is a *differentiable manifold* (think of an *open* subset of some Euclidean space). The dynamics F is then best viewed as a family of (nonautonomous) vector fields on (the manifold) M, parametrized by controls.

The control system Σ is **linear** if $U = \mathbb{R}^\ell$ and the dynamics $F : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^m$ has the form

$$F(t,x,u) = A(t)x + B(t)u$$

where $A(t) \in \mathbb{R}^{m\times m}$ and $B(t) \in \mathbb{R}^{m\times \ell}$ are matrices each of whose entries is a (continuous) function $\mathbb{R} \to \mathbb{R}$; that is, the dynamics F is *linear* in (x,u) for each fixed $t \in \mathbb{R}$.

One distinguishes controls of two types : *open* and *closed loop*. An **open loop control** can be basically an "arbitrary" function $u : [t_0, \infty) \to U$ for which the initial value problem (IVP)

$$\dot{x} = F(t,x,u)\,, \quad x(t_0) = x_0$$

has a well defined solution.

A **closed loop control** can be identified with a mapping $k : M \to U$ (which may depend on $t \geq t_0$) such that the initial value problem (IVP)

$$\dot{x} = F(t,x,k(x(\cdot)))\,, \quad x(t_0) = x_0$$

has a well defined solution. The mapping $k(\cdot)$ is called **feedback**.

One of the main aims of control theory is to find a strategy (input) such that the corresponding output has desired properties. Depending on the properties involved one gets more specific questions. Concepts like *controllability, observability, stabilizability, realization,* as well as *optimality* are fundamental in control theory.

Controllability

One say that a state $x_f \in \mathbb{R}^n$ is **reachable** from x_0 in time T if there exists an open loop control $u(\cdot)$ such that

$$x(0) = x_0 \quad \text{and} \quad x(T) = x_f.$$

If an arbitrary state x_f is reachable from an arbitrary state x_0 in time T, then the control system Σ is called **(completely) controllable**. In several situations one requires a weaker property of transfering an arbitrary state into a given one, in particular the origin. *A formulation of effective characterizations of controllable systems is an important task of control theory.*

Observability

In many situations of practical interest one observes not the state $x(\cdot)$ but its function $t \mapsto h(t, x(t))$, $t \geq t_0$. It is therefore often necessary to investigate the pair of equations (i.e. the *state equation* and the *observation equation*)

$$\begin{cases} \dot{x} = F(t, x, u) \\ \\ y = h(t, x). \end{cases}$$

This is a **control system with outputs**; that is, a 6-tuple

$$\Sigma = (M,\, U, \mathcal{U},\, F,\, \mathbb{R}^n,\, h).$$

In this case, $(M,\, U, \mathcal{U},\, F)$ is the *underlying* control system and h is the **measurement mapping**. We use the same symbol for the control system with outputs and its underlying control system. The mapping $h = (h_1, h_2, \ldots h_n) : \mathbb{R} \times M \to \mathbb{R}^n$ represents the vector of n *measurements (observations)*.

The control system with outputs Σ is **linear** if its underlying system is linear and the measurement mapping $h : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ is linear for each $t \in \mathbb{R}$.

This new system is said to be **(completely) observable** if, knowing a control $u(\cdot)$ and an *observation* $y(\cdot)$, on a given interval $[t_0, T]$, one can determine uniquely the initial condition x_0.

Stabilizability

Another important issue is that of stabilizability. Assume that for some $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in U$, $F(\bar{x}, \bar{u}) = 0$. A function $k : M \to U$ such that $k(\bar{x}) = \bar{u}$ is called a **stabilizing feedback** if $\bar{x}$ is a *stable equilibrium* for the system

$$\dot{x} = F(t, x, k(x(\cdot))).$$

In the theory of (ordinary) differential equations there exist several methods to determine whether a given equilibrium state is a stable one.

Realization

For a given initial condition $x_0 \in \mathbb{R}^n$, the control system with outputs

$$\begin{cases} \dot{x} = F(t, x, u), \quad x(t_0) = x_0 \\ \\ y = h(t, x) \end{cases}$$

defines a mapping which transforms open loops controls $u(\cdot)$ onto outputs

$$y(t) = h(t, x(t)), \quad t \in [t_0, T].$$

Denote this transformation by $\mathcal{R}$. What are its properties ? What conditions should a transformation $\mathcal{R}$ satisfy to be given by such a control system ? How, among all the possible "realizations" Σ of a transformation $\mathcal{R}$, do we find the simplest one ?

Optimality

Besides the above problems of structural character, in control theory one also asks *optimality questions*. In the so-called *time-optimal problem* one is looking for control which not only transfers a state x_0 onto x_f but does it in the minimal time T. More generally, one is looking for a control $u(\cdot)$ which *minimizes* a *functional* of the form

$$\mathcal{J} := \phi(x(T), T) + \int_{t_0}^{T} L(t, x, u) dt.$$

1.3 Examples

We shall mention several specific models of control systems.

Example 1. (Moving car) Suppose a car is to be driven along a straight road, and let its *distance* from an initial point 0 be $s(t)$ at time t. For simplicity, assume that the car is controlled only by the throttle, producing an *accelerating force* of $u_1(t)$ per unit mass, and by brake wich produces a *retarding force* of $u_2(t)$ per unit mass. Suppose that the only factors of interest are the car's position $x_1(t) := s(t)$ and velocity $x_2(t) := \dot{s}(t)$. Ignoring other forces such as road friction, wind resistance, etc. the equations which describe the state of the car at time t are

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = u_1 - u_2 \end{cases}$$

or, in matrix form,

$$\dot{x} = Ax + Bu(t)$$

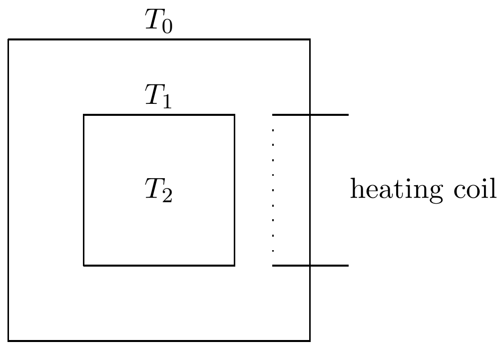
where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

It may be required to start from rest at 0 and reach some fixed point in the least possible time, or perhaps with minimum consumption of fuel. The mathematical problems are firstly to determine whether such objectives are achievable with the selected control variables, and if so, to find appropriate expressions for $u_1(\cdot)$ and $u_2(\cdot)$ as functions of time and/or $x_1(\cdot)$ and $x_2(\cdot)$.

Note : The complexity of the model could be increased so as to take into account factors such as engine speed and temperature, vehicle interior temperature, and so on.

Example 2. (Electrically heated oven) Let us consider a simple model of an ellectrically heated oven, which consists of a jacket with a coil directly heating the jacket and of an interior part.



Let T_0 denote the *outside temperature*. We make a simplifying assumption, that at an arbitrary moment $t \geq 0$, *temperature* in the *jacket* and in the *interior part* are uniformly distributed and equal to $T_1(t)$, $T_2(t)$. We assume also that the flow of heat through a surface is proportional to the area of the surface and to the difference of temperature between the separated media. Let $u(t)$ be the *intensity of the heat input* produced by the coil at moment $t \geq 0$. Let moreover a_1, a_2 denote the *area* of exterior and interior surfaces of the jacket, respectively, c_1, c_2 denote *heat capacities* of the jacket and the interior of the oven, respectively, and r_1, r_2 denote *radiation* coefficients of the exterior and interior surfaces of the jacket, respectively. An increase of heat in the jacket is equal to the amount of heat produced by the coil reduced by the amount of heat which entered the interior and exterior of the oven. Therefore, for the interval $[t, t+\Delta t]$, we have the following balance :

$$c_1\left(T_1(t+\Delta t) - T_1(t)\right) \approx u(t)\Delta t - \left(T_1(t) - T_2(t)\right) a_1 r_1 \Delta t - \left(T_1(t) - T_0\right) a_2 r_2 \Delta t\,.$$

Similarly, an increase of heat in the interior of the oven is equal to the amount of heat radiated by the jacket :

$$c_2\left(T_2(t+\Delta t) - T_2(t)\right) = \left(T_1(t) - T_2(t)\right) a_1 r_2 \Delta t\,.$$

Dividing the obtained identities by Δt and taking the limit, as $\Delta \to 0$, we

obtain :

$$\begin{cases} c_1\dot{T}_1 = u - (T_1 - T_2)a_1r_1 - (T_1 - T_0)a_2r_2 \quad \text{(for the jacket)} \\ \\ c_2\dot{T}_2 = (T_1 - T_2)a_1r_1 \quad \text{(for the oven interior).} \end{cases}$$

Les us notice that, according to the physical interpretation, $u(t) \geq 0$ for $t \geq 0$. Let the *state variables* be the excesses of temperature over the exterior, that is

$$x_1 := T_1 - T_0 \quad \text{and} \quad x_2 := T_2 - T_0.$$

Then we can write the equations above in matrix form, namely

$$\dot{x} = Ax + Bu(t)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u \end{bmatrix}, \quad A = \begin{bmatrix} -\dfrac{a_1r_1 + a_2r_2}{c_1} & \dfrac{a_1r_1}{c_1} \\ \\ \dfrac{a_1r_1}{c_2} & -\dfrac{a_1r_1}{c_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{c_1} \\ \\ 0 \end{bmatrix}.$$

It is natural to limit the considerations to the case when $x_1(0) \geq 0$ and $x_2(0) \geq 0$. It is physically obvious that if $u(t) \geq 0$ for $t \geq 0$, then also $x_1(t) \geq 0$, $x_2(t) \geq 0$ for $t \geq 0$.

Two interesting aspects to be discussed are firstly whether it is possible to maintain the temperature of the oven interior at any desired level merely by altering u, and secondly, to determine whether the value of T_2 can be determined even if it is not possible to measure it directly.

NOTE : If the desired objective is attainable, then there may well be many different suitable control schemes, and considerations of economy, practicability of application, and so on will then determine how control is actually applied.

Example 3. (Controlled environment) Consider a controlled environment consisting of *rabbits* and *foxes*, the number of each at time t being $x_1(t)$ and

$x_2(t)$, respectively. Suppose that without the presence of foxes the number of rabbits would grow exponentially, but that the rate of growth of rabbit population is reduced by an amount proportional to the number of foxes. Furthermore, suppose that, without rabbits to eat, the fox population would decrease exponentially, but the rate of growth in the number of foxes is increased by an amount proportional to the number of rabbits present. Under these assumptions, the system of equations can be written

$$\begin{cases} \dot{x}_1 = a_1 x_1 - a_2 x_2 \\ \\ \dot{x}_2 = a_3 x_1 - a_4 x_2 \end{cases}$$

where a_1, a_2, a_3 and a_4 are positive constants.

Example 4. (Satellite problem) We shall consider a point mass in an inverse square law force field. The motion of a unit mass is governed by a pair of second order equations in the radius r and the angle θ (polar coordinates). If we assume that the unit mass (say a satellite) has the capability of thrusting in the *radial direction* with the *thrust* $u_1(\cdot)$ and thrusting in the *tangential direction* with *thrust* $u_2(\cdot)$, then we have

$$\begin{cases} \ddot{r} = r\dot{\theta}^2 - \dfrac{k}{r^2} + u_1(t) \\ \\ \ddot{\theta} = -\dfrac{2\dot{\theta}\dot{r}}{r} + \dfrac{1}{r} u_2(t). \end{cases}$$

If $u_1(t) = u_2(t) = 0$, these equations admit the solution

$$r(t) = \sigma \quad (\sigma \text{ constant}) \quad \text{and} \quad \theta(t) = \omega t \quad (\omega \text{ constant})\,; \quad \sigma^3\omega^2 = k\,.$$

That is, *circular orbits are possible.* If we let x_1, x_2, x_3, and x_4 be given by

$$x_1 := r - \sigma, \quad x_2 := \dot{r}, \quad x_3 := \sigma(\theta - \omega t), \quad x_4 := \sigma(\dot{\theta} - \omega)$$

and normalize σ to 1, then it is easy to see that the *linearized equations* of motion about the given solution are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Example 5. (Market economy) Suppose that the *sales* $S(t)$ of a product are affected by the *amount of advertising* $A(t)$ in such a way that the rate of change of sales decreases by an amount proportional to the advertising applied to the share of the market not already purchasing the product. If the total extent of the market is M, the *state equation* is therefore

$$\dot{S} = -aS + bA(t)\left(1 - \frac{S}{M}\right)$$

subject to

$$S(0) = S_0$$

where a and b are positive constants. In practice, the amount of advertising will be limited (that is, $0 \leq A(t) \leq K$, where K is a constant), and the aim would be to find the advertising schedule (that is, the function $A(t)$ which maximizes the sales over some period of time).

Example 6. (Soft landing) Let us consider a spacecraft of *total mass* M moving vertically with the gas thruster directed toward the landing surface. Let $h(\cdot)$ be the *height* of the spacecraft above the surface, $u(\cdot)$ the *thrust* of its engine produced by the expulsion of gas from the jet. The gas is a product of the combustion of the fuel. The combustion decreases the total mass of the spacecraft, and the thrust u is proportionalto the speed with which the mass decreases. Assuming that there is no atmosphere above the surface and that

g is gravitational acceleration, one arrives at the following equations :

$$\begin{cases} M\ddot{h} &= -gM + u(t) \\ \\ \dot{M} &= -ku(t) \quad (k > 0) \end{cases}$$

with the initial conditions

$$M(0) = M_0\,, \quad h(0) = h_0\,, \quad \dot{h}(0) = h_1.$$

One imposes additional constraints on the control parameter of the type

$$0 \le u \le \alpha \quad \text{and} \quad M \ge m,$$

where m is the mass of the spacecraft without fuel. Let us fix $T > 0$. The *soft landing problem* consists of finding a control $u(\cdot)$ such that for the solutions $M(\cdot)$, $h(\cdot)$ of the above equations

$$M(t) \ge m\,, \quad h(t) \ge 0\,, \quad t \in [0, T]\,, \quad \text{and} \quad h(T) = \dot{h}(T) = 0.$$

NOTE : A natural *optimization* question arises when the moment T is not fixed and one is minimizing the landing time.

1.4 Matrix Theory (review)

Matrices and determinants

We write a **matrix** as follows

$$A = \begin{bmatrix} a_{ij} \end{bmatrix} \quad (i = 1, 2, \ldots, m\,; \; j = 1, 2, \ldots, n)$$

where a_{ij} is the element (entry) in its i^{th} row and j^{th} column, and A thus has m **rows** and n **columns**; we use to say that A is an $m \times n$ matrix. We shall denote by $\mathbb{R}^{m\times n}$ the set (vector space) of all $m \times n$ matrices with real entries.

NOTE : (1) It is convenient to *identify* the set (vector space) $\mathbb{R}^m$ of all *m-tuples* of real numbers with the set (vector space) $\mathbb{R}^{m\times 1}$ of all *column m-matrices* (or column m-vectors).

(2) There is a natural one-to-one correspondence between $m\times n$ matrices and linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$: the vector spaces $\mathbb{R}^{m\times n}$ and $L(\mathbb{R}^n, \mathbb{R}^m)$ are *isomorphic.*

The **transpose** A^T of A is obtained by interchanging the rows and columns of $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, so $A^T := \begin{bmatrix} a_{ji} \end{bmatrix}$ is an $n \times m$ matrix. If λ is a *scalar* (real number), and A and B are matrices of appropriate size, then :

(a) $(A^T)^T = A$.

(b) $(A+B)^T = A^T + B^T$.

(c) $(\lambda A)^T = \lambda A^T$.

(d) $(AB)^T = B^T A^T$.

If A is *invertible*, then A^T is invertible, too and $(A^T)^{-1} = (A^{-1})^T$.

We can write a matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{m\times n}$ in the following forms :

- $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ with $a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^{m\times 1} \quad (j = 1, 2, \dots, n)$

- $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ with $a_i = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \in \mathbb{R}^{1\times n} \quad (i = 1, 2, \dots, m)$.

When $m = n$, A is said to be **square** of order n. We shall write

$$I_n = \begin{bmatrix} \delta_{ij} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

for the **unit matrix** (or **identity matrix**) of order n. Here, δ_{ij} stands for **Kronecker**'s **symbol**; that is,

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The *unit matrix* I_n has all its elements zero except those on the *main diagonal*; any (square) matrix of this form is called a **diagonal matrix**, written

$$\operatorname{diag}\left(a_{11},\, a_{22},\, \ldots,\, a_{nn}\right).$$

Two $n \times n$ matrices A and B related by

$$B = S^{-1}AS$$

are called **similar**. This is a basic *equivalence relation* on matrices.

NOTE : Two matrices are similar if and only if they represent the same linear mapping in different bases. *Any matrix property that is preserved under similarity is a property of the underlying linear mapping.*

If $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$, then its **trace** is the sum of all elements on the main diagonal; that is,

$$\operatorname{tr}(A) := \sum_{i=1}^{n} a_{ii}.$$

The trace operator has some important properties :

(a) $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr}(A)$, where λ is a scalar.

(b) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

(c) $\operatorname{tr}(I_n) = n$.

(d) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(e) $\operatorname{tr}(A^T) = \operatorname{tr}(A)$.

(f) $\operatorname{tr}(A^T A) \geq 0$.

Exercise 1 Given $A, S \in \mathbb{R}^{n\times n}$ with S invertible, show that

$$\text{tr}\,\left(SAS^{-1}\right) = \text{tr}\,(A).$$

That is, *similar matrices have the same trace.*

We recall briefly the main properties of the **determinant** function

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}\,, \quad A \mapsto \det\,(A).$$

These are :

(a) $\det\,(AB) = \det\,(A)\cdot\det\,(B)$.

(b) $\det\,(I_n) = 1$.

(c) $\det\,(A) \neq 0$ if and only if A is invertible.

NOTE : There is a *unique* function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ having these three properties. For $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$ we have

$$\det\,(A) = \sum_{\alpha} \text{sgn}(\alpha)\; a_{1\alpha(1)}a_{2\alpha(2)}\cdots a_{n\alpha(n)}$$

where the sum is taken over the $n!$ *permutations* (on n elements) $\alpha \in S_n$.

Exercise 2 Given $A, S \in \mathbb{R}^{n\times n}$ with S invertible, show that

$$\det\,\left(SAS^{-1}\right) = \det\,(A).$$

That is, *similar matrices have the same determinant.*

If $\det\,(A) = 0$, A is **singular**, otherwise **nonsingular** (or invertible); in the latter case, the **inverse** of A is

$$A^{-1} = \frac{1}{\det\,(A)}\,\text{adj}\,(A)$$

where $\text{adj}\,(A) := \left[A_{ij}\right]^T$ is the **adjoint** of A; here,

$$A^i_j := (-1)^{i+j} M_{ij} \quad (\,\text{the } \textbf{cofactor} \text{ of } a_{ij}\,)$$

and M_{ij} is the determinant of the submatrix formed by deleting the i^{th} row and the j^{th} column of A.

NOTE : Let $A \in \mathbb{R}^{n\times n}$. Then $\det(A) = 0$ (the matrix A is singular) if and only if $Ax = 0$ for some *nonzero* column n-vector $x \in \mathbb{R}^{n\times 1}$.

Linear dependence and rank

Consider a set of column m-vectors

$$a_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad a_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad a_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, then the vector

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = \begin{bmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn} \end{bmatrix}$$

is called a **linear combination** of $a_1, a_2, \ldots, a_n$. If there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

then the vectors $a_1, a_2, \ldots, a_n$ are said to be **linearly dependent**; otherwise, they are **linearly independent**.

NOTE : We can equally well consider row n-vectors.

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{m \times n}$. The **rank** of A, denoted by $\text{rank}\,(A)$, is defined as the maximum number of linearly independent columns (or rows) of A. Clearly, $\text{rank}\,(A) \leq \min\{m, n\}$. Consider the **kernel** (or **null-space**)

$$\ker\,(A) : = \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$$

and the **image space** (or **column space**)

$$\text{im}\,(A) : = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

The *dimension* of $\ker\,(A)$ is termed the **nullity** of A. $\text{im}\,(A)$ is nothing more than the (vector) space *spanned* by the columns of A; that is,

$$\text{im}\,(A) = \text{span}\,\{a_1, a_2, \ldots, a_n\} : = \{\alpha_1 a_1 + \alpha_2 a_2 \cdots + \alpha_n a_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{R}\}$$

where $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$. Hence *the dimension of* $\text{im}\,(A)$ *is equal to* $\text{rank}\,(A)$.

NOTE : $\text{im}\,(A^T)$ is also known as the *row space* of A; it is the (vector) space spanned by the rows of A (i.e. the columns of A^T).

An important result states that (for a matrix $A \in \mathbb{R}^{m \times n}$) :

$$\boxed{\text{rank}\,(A) + \dim \ker\,(A) = n.}$$

Rank is invariant under multiplication by a nonsingular matrix. In particular, *rank is invariant under similarity.* However, multiplication by rectangular or singular matrices can alter the rank, and the following formula shows exactly how much alteration occurs.

If $A \in \mathbb{R}^{m \times n}$ *and* $B \in \mathbb{R}^{n \times p}$*, then :*

$$\boxed{\text{rank}\,(AB) = \text{rank}\,(B) - \dim \ker\,(A) \cap \text{im}\,(B).}$$

Exercise 3 Given $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, show that :

(a) $\text{rank}\,(AB) \leq \min\{\text{rank}\,(A), \, \text{rank}\,(B)\}$.

(b) $\text{rank}\,(A) + \text{rank}\,(B) - n \leq \text{rank}\,(AB)$.

Suppose now that $a_{ij} \quad (i = 1, 2, \ldots, m\,; \; j = 1, 2, \ldots, n)$ are the coefficients in a set of m linear *algebraic equations* in n unknowns

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \,, \quad i = 1, 2, \ldots, m.$$

These equations can be written in matrix form as

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Such a linear system (of equations) possesses a solution if and only if

$$\operatorname{rank}(A) = \operatorname{rank} \begin{bmatrix} A & b \end{bmatrix}$$

where $\begin{bmatrix} A & b \end{bmatrix}$ is the $m \times (n+1)$ matrix obtained by appending b to A as an extra column.

Two particular cases should be mentioned :

- When $A \in \mathbb{R}^{n \times n}$, the linear system $Ax = b$ has a unique solution if and only if A is nonsingular.
- When $A \in \mathbb{R}^{m \times n}$, the *homogeneous* linear system $Ax = 0$ has a nonzero solution if and only if $\operatorname{rank}(A) < n$.

NOTE : Similar remarks apply to the set of equations

$$yA = c$$

where

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad y = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}.$$

Eigenvalues and eigenvectors

Let $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$. A *nonzero vector* $w \in \mathbb{R}^{n\times 1}$ is called an **eigenvector** (or **characteristic vector**) of A if there is a *scalar* (real number) λ such that

$$Aw = \lambda w.$$

The scalar λ is called the **eigenvalue** (or **characteristic value**) associated with the eigenvector w. Geometrically, $Aw = \lambda w$ says that under the (linear) mapping $x \mapsto Ax$ the eigenvectors experience only changes in magnitude or sign. The eigenvalue λ is simply the amount of "stretch" or "shrink" to which the eigenvector w is subjected when acted upon by A.

Note : The words *eigenvalue* and *eigenvector* are derived from the German word *eigen*, which means "owen by" or "peculiar to".

The set of *distinct* eigenvalues, denoted by $\sigma(A)$, is called the **spectrum** of A. We have

$$\lambda \in \sigma(A) \iff \lambda I_n - A \text{ is singular} \iff \det(\lambda I_n - A) = 0.$$

The set (vector space) of all eigenvectors with eigenvalue λ, together with the zero vector, is called the λ-**eigenspace** of A and is denoted by E_λ. That is,

$$E_\lambda := \ker(\lambda I_n - A).$$

The eigenvectors $w \in E_\lambda$ are found by solving the equation

$$(\lambda I_n - A)\, w = 0.$$

This matrix equation is equivalent to a system of n linear algebraic equations; the solution space is exactly the λ-eigenspace E_λ.

Exercise 4 Let $A \in \mathbb{R}^{n\times n}$. If $\lambda_1, \lambda_2, \dots, \lambda_r$ are r $(r \leq n)$ *distinct* eigenvalues of A with corresponding eigenvectors $w_1, w_2, \dots, w_r$, show that the vectors $w_1, w_2, \dots, w_r$ are *linearly independent.*

The **characteristic polynomial** of $A \in \mathbb{R}^{n\times n}$ is

$$\mathrm{char}_A(\lambda) := \det(\lambda I_n - A).$$

The algebraic equation

$$\mathrm{char}_A(\lambda) \equiv \lambda^n + k_1\lambda^{n-1} + \cdots + k_{n-1}\lambda + k_n = 0$$

is called the **characteristic equation** of A.

NOTE : The *degree* of the characteristic polynomial (equation) is n and the leading term is λ^n. The eigenvalues of A are exactly the real roots of $\mathrm{char}_A(\lambda)$.

The *fundamental theorem of algebra* insures that the characteristic polynomial $\mathrm{char}_A(\lambda)$ has n roots, but some roots may be complex numbers, and some roots may be repeated. A complex root of the characteristic polynomial is called a **complex eigenvalue** of A. The complex eigenvalues must occur in conjugate pairs. If λ is a complex eigenvalue of the matrix A, we write $\lambda \in \sigma_{\mathbb{C}}(A)$. Henceforth, we shall refer to both sets $\sigma(A)$ and $\sigma_{\mathbb{C}}(A)$ as the spectrum of A. (In fact, $\sigma_{\mathbb{C}}(A)$ is the spectrum of the "complexification" of (the linear mapping) $A : \xi + i\,\eta \mapsto A\xi + iA\eta$).

An important result is the following : *If the matrix A is symmetric (i.e. $A = A^T$), then all its eigenvalues are real.*

A useful result is the CAYLEY-HAMILTON THEOREM, which states that *every square matrix satisfies its own characteristic equation*; that is, if $A \in \mathbb{R}^{n\times n}$, then :

$$\mathrm{char}_A(A) \equiv A^n + k_1A^{n-1} + \cdots + k_{n-1}A + k_nI_n = O.$$

Let $\lambda \in \sigma_{\mathbb{C}}(A)$.

- The **algebraic multiplicity** m_λ of λ is the number of times it is repeated as a root of the characteristic polynomial.

- The **geometric multiplicity** d_λ of λ is the dimension of the λ-eigenspace E_λ. In other words, d_λ is the maximum number of linearly independent eigenvectors associated with λ.

In general, $d_\lambda \leq m_\lambda$. The following remarkable result holds :

The matrix $A \in \mathbb{R}^{n\times n}$ *is diagonalizable (that is,* A *is similar to a diagonal matrix) if and only if* $d_\lambda = m_\lambda$. If all the eigenvalues of A are real and distinct, then A is diagonalizable. The converse is *not* true.

Exercise 5 Let $A \in \mathbb{R}^{n\times n}$ with complex eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (listed with their algebraic multiplicities). Show that :

(a) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{tr}(A)$.

(b) $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A)$.

Quadratic forms

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{n\times n}$ be a *symmetric* matrix. A function $q : \mathbb{R}^n = \mathbb{R}^{n\times 1} \to \mathbb{R}$ defined by

$$\begin{aligned} q(x) \quad &:= \quad x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \\ &= \quad a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots \end{aligned}$$

is called a **quadratic form** (on $\mathbb{R}^n$). Clearly, $q(0) = 0$.

A quadratic form q is said to be :

(a) **positive definite** provided $q(x) > 0$ for all *nonzero* $x \in \mathbb{R}^{n\times 1}$.

(b) **negative definite** provided $q(x) < 0$ for all *nonzero* $x \in \mathbb{R}^{n\times 1}$.

(c) **positive semi-definite** provided $q(x) \geq 0$ for all $x \in \mathbb{R}^{n\times 1}$.

(d) **negative semi-definite** provided $q(x) \leq 0$ for all $x \in \mathbb{R}^{n\times 1}$.

Finally, we call q **indefinite** provided q takes positive as well as negative values.

NOTE : (1) The terms describing the quadratic form q are also applied to the (symmetric) matrix A associated with the form.

(2) The definitions on *definiteness* and *semi-definiteness* can be extended to scalar functions (defined on some $\mathbb{R}^n$) which are not necessarily quadratic.

One simple way of determining the sign properties of a quadratic form is the following:

The quadratic form $q : x \mapsto x^T Ax$ *(or, equivalently, the matrix* A *) is :*

- *positive definite if and only if all the eigenvalues of* A *are positive.*
- *negative definite if and only if all the eigenvalues of* A *are negative.*
- *positive semi-definite if and only if all the eigenvalues of* A *are nonnegative.*
- *negative semi-definite if and only if all the eigenvalues of* A *are nonpositive.*

An alternative approach involves the **principal minors** P_i of A, these being any i^{th} order *minors* whose main diagonal is part of the main diagonal of A. In particular, the **leading principal minors** of A are

$$\Delta_1 := a_{11}\,, \quad \Delta_2 := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\,, \quad \Delta_3 := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\,, \quad \text{etc.}$$

The SYLVESTER CONDITIONS state that *the quadratic form* $q : x \mapsto x^T Ax$ *(or, equivalently, the matrix* A *) is* :

- *positive definite if and only if* $\Delta_i > 0\,, \quad i = 1, 2, \ldots, n$;
- *negative definite if and only if* $(-1)^i \Delta_i > 0\,, \quad i = 1, 1, \ldots, n$;
- *positive semi-definite if and only if* $P_i \geq 0$ *for all principal minors*;

- *negative semi-definite if and only if* $(-1)^i P_i \geq 0$ *for all principal minors.*

If q *satisfies none of the above conditions, then it is indefinite.*

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix such that $\operatorname{rank}(A) = r$. Then A is positive semi-definite ($A^T = A \geq 0$) if and only if $A = B^T B$ for some matrix B with $\operatorname{rank}(B) = r$.

The matrix exponential

In order to define the *exponential* of a matrix, we need to discuss the *convergence* of (infinite) sequences and series involving matrices.

Because $\mathbb{R}^{m\times n}$ is a vector space of dimension mn, magnitudes of matrices $A \in \mathbb{R}^{m\times n}$ can be "measured" by employing any *norm* on $\mathbb{R}^{mn}$. One of the simplest *matrix norms* is the following :

$$\|A\| := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)} \quad \text{for } A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

For a column matrix (vector) $x \in \mathbb{R}^{m\times 1}$ this gives the **Euclidean norm**

$$\|x\|_e := \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}.$$

Note : (1) Other matrix norms can also be defined. For example, *any* norm $\|\cdot\|_*$ that is defined on $\mathbb{R}^m$ and $\mathbb{R}^n$ *induces* a matrix norm on $\mathbb{R}^{m\times n}$ by setting

$$\|A\|_* := \max_{\|x\|_*=1} \|Ax\|_* \quad \text{for } A \in \mathbb{R}^{m\times n} \text{ and } x \in \mathbb{R}^{n\times 1}.$$

(2) The matrix norm $\|\cdot\|$ and the Euclidean norm $\|\cdot\|_e$ are *compatible* :

$$\|Ax\|_e \leq \|A\|\,\|x\|_e.$$

The matrix norm has all the usual properties of a *norm*; that is (for $A, B \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$) :

(a) $\|A\| \geq 0\,, \quad \text{and} \quad \|A\| = 0 \iff A = 0.$

(b) $\|\lambda A\| = |\lambda|\|A\|.$

(c) $\|A + B\| \leq \|A\| + \|B\|.$

Also, the following two relations hold (provided that the product AB is defined) :

(d) $\|AB\| \leq \|A\|\,\|B\|.$

(e) $\|A^k\| \leq \|A\|^k\,, \quad k = 0, 1, 2, \ldots.$

A *sequence* $(A_k)_{k\in\mathbb{N}}$ of matrices in $\mathbb{R}^{m\times n}$ is said to **converge** to the *limit* $A \in \mathbb{R}^{m\times n}$, denoted by

$$\lim_{k\to\infty} A_k = A,$$

if the sequence $(\|A_k - A\|)_{k\in\mathbb{N}}$ of (positive) real numbers converges to 0; that is, for every $\varepsilon > 0$ there exists an $\nu \in \mathbb{N}$ such that for $k \geq \nu$, $\|A - A_k\| < \varepsilon$.

A necessary and sufficient condition for convergence is that each entry of A_k tends (converges) to the corresponding entry of A as $k \to \infty$.

The (infinite) matrix *series* $\sum\limits_{k\geq 0} A_k$ **converges** provided the sequence of *partial sums* $(S_k)_{k\in\mathbb{N}}$, where $S_k := A_1 + A_2 + \cdots + A_k$, converges to a limit S (as $k \to \infty$); if a limit exists, then it is unique and we shall write $S = \sum\limits_{k=0}^{\infty} A_k$. The series is **absolutely convergent** if the scalar series $\sum\limits_{k\geq 0} \|A_k\|$ is convergent; *an absolutely convergent matrix series is convergent.*

Consider now a matrix $A \in \mathbb{R}^{n\times n}$.

Exercise 6 Show that the matrix power series $\sum\limits_{k\geq 0} \dfrac{t^k}{k!}A^k$ is *convergent* (in fact, absolutely convergent) for every $t \in \mathbb{R}$.

We define the **matrix exponential** of A by

$$\boxed{\exp(tA) := I_n + tA + \tfrac{t^2}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k.}$$

The matrix exponential has a number of important properties.

(a) $\dfrac{d}{dt}\left(\exp(tA)\right) = A\exp(tA) = \exp(tA)A.$

(b) $\exp\left((t+s)A\right) = \exp(tA)\cdot\exp(sA).$

(c) $\exp(A) = \lim_{k\to\infty}\left(I_n + \dfrac{A}{k}\right)^k.$

(d) $\det\left(\exp(A)\right) = e^{\operatorname{tr}(A)}.$

From (b) it follows that

$$\exp(tA)\cdot\exp(-tA) = I_n$$

and thus

$$\boxed{\exp(tA)^{-1} = \exp(-tA).}$$

1.5 Exercises

Exercise 7 Suppose $A, S \in \mathbb{R}^{n\times n}$ and S is invertible. Show that

$$(S^{-1}AS)^2 = S^{-1}A^2S.$$

Generalize to $(S^{-1}AS)^n$.

Exercise 8 Set

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad n \geq 2.$$

Write $A = uu^T \in \mathbb{R}^{n\times n}$ and show that A is singular.

Exercise 9 Let $A \in \mathbb{R}^{n\times n}$ and $0 \neq b \in \mathbb{R}^{n\times 1}$ such that

$$A^r b \neq 0\,, \quad A^{r+1}b = 0$$

for some positive integer $r < n$. By considering the equation

$$c_0 b + c_1 Ab + c_2 A^2 b + \cdots + c_r A^r b = 0$$

where the coefficients $c_i \in \mathbb{R}$, deduce that the (column) vectors

$$b,\ Ab,\ A^2b,\ \cdots,\ A^rb$$

are linearly independent.

Exercise 10 Verify that

$$\operatorname{rank}(A^TA) = \operatorname{rank}(A) = \operatorname{rank}(AA^T)$$

for the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{bmatrix} \in \mathbb{R}^{3\times 4}.$$

Exercise 11 Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for each given matrix.

(a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

(c) $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$.

(d) $\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & -1 \\ ab & a+b \end{bmatrix}$.

(f) $\begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(h) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(i) $\begin{bmatrix} -1 & 3 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

(j) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Exercise 12 Let $A \in \mathbb{R}^{n\times n}$.

(a) How are the eigenvalues of $A - \mu I_n$ related to those of A ?

(b) How are the eigenvalues of μA related to those of A ?

(c) How are the eigenvalues of A^n related to those of A ?

(d) How are the eigenvalues of A^{-1} related to those of A ?

Exercise 13 Consider a matrix $A \in \mathbb{R}^{n\times n}$. Show that :

(a) The characteristic polynomials of A and A^T are the same.

(b) The characteristic polynomials of A and $S^{-1}AS$ are the same.

(c) If $n = 2$, the characteristic polynomial of A can be written as follows

$$\mathrm{char}_A(\lambda) = \lambda^2 - \mathrm{tr}\,(A)\lambda + \det\,(A).$$

Exercise 14 Let the matrix $A \in \mathbb{R}^{n\times n}$ be invertible and w an eigenvector of A with associated eigenvalue λ.

(a) Is w an eigenvector of A^3 ? If so, what is the eigenvalue ?

(b) Is w an eigenvector of A^{-1} ? If so, what is the eigenvalue ?

(c) Is w an eigenvector of $A + 2I_n$? If so, what is the eigenvalue ?

(d) Is w an eigenvector of $7A$? If so, what is the eigenvalue ?

Exercise 15

(a) A **skew-symmetric** matrix $S \in \mathbb{R}^{n\times n}$ is defined by $S^T = -S$. If $q = x^T S x$ show (by considering q^T) that $q = 0$ for all (column) vectors $x \in \mathbb{R}^{n\times 1}$.

(b) Show that any matrix $A \in \mathbb{R}^{n\times n}$ can be written as $A = A_1 + A_2$, where A_1 is symmetric and A_2 is skew-symmetric. Hence (using the result of (a)) deduce that

$$x^T A x = x^T A_1 x \quad \text{for all (column) vectors } x \in \mathbb{R}^{n\times 1}.$$

Exercise 16 Prove that

$$\exp((t+s)A) = \exp(tA)\exp(sA).$$

[Hint: Multiply the series in powers of A formally; the legitimacy of the term-by-term multiplication is assured by the fact that $\exp(tA)$ is absolutly convergent.]

Exercise 17

(a) Show that $\exp(tA)\cdot\exp(tB)$ does not have to be either $\exp\left(t(A+B)\right)$ or $\exp(tB)\cdot\exp(tA)$ by calculating all three, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) Suppose that $AB = BA$. Show that

$$\exp\left(t(A+B)\right) = \exp(tA)\cdot\exp(tB) = \exp(tB)\cdot\exp(tA).$$

[Hint: Show that if $P(t) = \exp\left(t(A+B)\right)\cdot\exp(-tA)\cdot\exp(-tB)$, then $\dot{P}(t) = 0$ for all t. Since $P(0) = I_n$, we must have $P(t) = I_n$.]

Exercise 18

(a) Find $\exp(tA)$ if

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Generalize to $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ (the *diagonal matrix* with diagonal elements $a_1, a_2, \ldots, a_n$).

(b) Consider the matrix

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

Show that

$$\exp(tA) = \begin{bmatrix} e^{at} & bte^{at} \\ 0 & e^{at} \end{bmatrix}.$$

(c) A matrix A is **nilpotent** if some power A^k is the zero matrix. Then the matrix exponential $\exp(tA)$ can be calculated easily because the series stops with the power A^{k-1}. That is, we have $A^k = A^{k+1} = \cdots = 0$, so

$$\exp(tA) = I_n + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1}\,.$$

Find $\exp(tA)$ for

i. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

ii. $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix}$.

Exercise 19 Find $\exp(tA)$, where A is given.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(f) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(g) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

Exercise 20 TRUE or FALSE ? Motivate your answers.

(a) If $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$, then

$$\det(\lambda A) = \lambda \det(A).$$

(b) If $A,\ B \in \mathbb{R}^{n\times n}$ then

$$\det(A+B) = \det(A) + \det(B).$$

(c) If $A \in \mathbb{R}^{n\times n}$ then

$$\det(AA^T) = \det(A^T A).$$

(d) If $A \in \mathbb{R}^{m\times n}$ then

$$\operatorname{rank}(A^T A) = \operatorname{rank}(A) = \operatorname{rank}(AA^T).$$

(e) A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if 0 is *not* an eigenvalue of A.

(f) If $t \in \mathbb{R}$ then

$$\exp\left(t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

(g) If $A,\ B \in \mathbb{R}^{n\times n}$ then

$$\exp(A+B) = \exp(A)\cdot\exp(B).$$

(h) If $A \in \mathbb{R}^{n\times n}$ then

$$\det\ (\exp(A)) \neq 0.$$