Control Systems on the Oscillator Group

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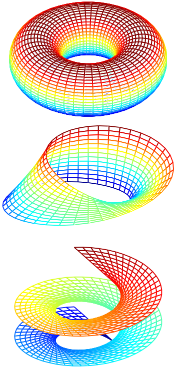
Outline

1. Background
   - Smooth manifolds
   - Lie groups
   - Left invariant control systems
2. Control systems on $\mathfrak{h}_3^\diamond$
   - The Oscillator group and its structure
   - Automorphisms of the Oscillator group
   - Classification of LiCAS systems

Visually in $\mathbb{R}^3$

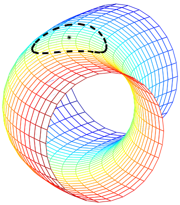


- Smooth surfaces in $\mathbb{R}^3$.
- Generally not globally diffeomorphic to $\mathbb{R}^2$.
- Locally diffeomorphic to $\mathbb{R}^2$.
- In general: smooth manifolds are "higher dimensional smooth surfaces".

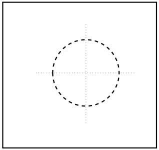
Coordinate chart

Definition

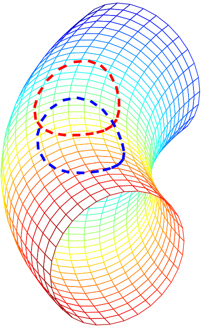
Let M be some set. Given $U \subseteq$ M, a injection $\varphi : U \to \mathbb{R}^n$ with open image, the pair (U, φ) is called a chart for M.



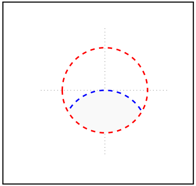
$$\varphi : U \to \mathbb{R}^n$$



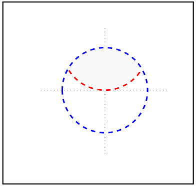
Compatible charts



$$\varphi : U \to \mathbb{R}^n$$



$$\phi : V \to \mathbb{R}^n$$



Smooth manifold

Definition

We call M a smooth manifold if the following hold:

1. It is covered by a collection of charts.
2. M has an atlas; that is: M can be written as a union of compatible charts.

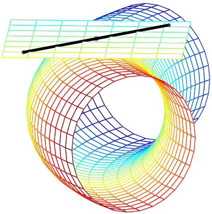
- Smooth manifolds are locally diffeomorphic to $\mathbb{R}^n$, ($n = \dim \mathrm{M}$).
- Generalised space on which calculus can be done.

Tangent vectors

Definition

A tangent vector at a point $m \in \mathrm{M}$ is an equivalence class of curves: $t \mapsto c_1(t) \sim t \mapsto c_2(t)$ at m iff $c_1(0) = c_2(0) = m$ and

$$\frac{d}{dt}(\varphi \circ c_1)(0) = \frac{d}{dt}(\varphi \circ c_2)(0)$$



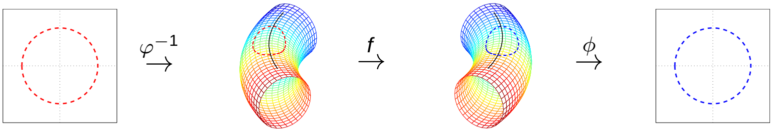
- The set of tangent vectors to M at m forms a vector space. It is denoted $T_m\mathrm{M}$ and is called the tangent space to M at $m \in \mathrm{M}$.
- The tangent bundle of M, denoted by $T\mathrm{M}$, is the disjoint union of the tangent spaces to M. That is $T\mathrm{M} = \cup_{m \in M} T_m\mathrm{M}$.

Tangent map

Definition

A map $f : \mathrm{M} \to \mathrm{N}$ is smooth if it's smooth in local coordinates. The tangent map of f, $Tf : T\mathrm{M} \to T\mathrm{N}$ is given by

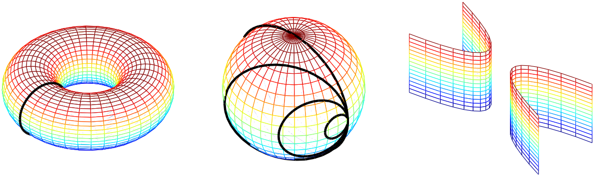
$$T_m f \cdot \dot{c}(0) = \left.\frac{d}{dt} f(c(t))\right|_{t=0}$$



- Note that $T_m f$ is a linear map.

Connected and simply connected

- A manifold M is connected if for any two points $m_0, m_1 \in \mathsf{M}$ there exists a continuous path connecting m_0 and m_1.
- A manifold M is simply connected if any continuous loop may be (continuously) contracted into a point.



Lie groups

Definition

A (real) Lie group is a group G equipped with the structure of a smooth manifold (over $\mathbb{R}$), such that the group product

$$\mu : \mathsf{G} \times \mathsf{G} \to \mathsf{G}, \ (g_1, g_2) \mapsto g_1 \, g_2$$

is smooth.

- Group inversion $g \mapsto g^{-1}$ is a diffeomorphism.
- Left translations $L_g : \mathsf{G} \to \mathsf{G}, \ h \mapsto g\,h$ are diffeomorphisms.

Examples

- $\mathbb{R}^n$ with ordinary vector addition as the group operation.
- The group $\mathrm{GL}(n, \mathbb{R})$ of invertible matrices of order n over the field $\mathbb{R}$ (smooth structure as open subset inherited from $\mathbb{R}^{n^2}$).
- Any closed subgroup of a real Lie group.
- The orthogonal group $\mathrm{O}_n(\mathbb{R})$, consisting of all $n \times n$ orthogonal matrices with real entries.
- The Euclidean group $\mathrm{E}_n(\mathbb{R})$ is the Lie group of all Euclidean motions, i.e., isometric affine maps, of n-dimensional Euclidean space $\mathbb{R}^n$.

Lie (or tangent) algebra

Definition

A real Lie algebra $\mathfrak{g}$ is a vector space over $\mathbb{R}$ together with a bilinear skew-symmetric binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \quad \text{for all } A, B, C \in \mathfrak{g}.$$

Examples:

- Any real vector space with $[\cdot,\cdot] = 0$;
- $\mathbb{R}^3$ with $[v,w] = v \times w$ the cross product;
- $\mathbb{R}^{n\times n}$ with $[A,B] = AB - BA$.

The Lie functor

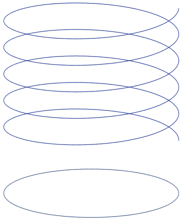
- The vector space $T_{\mathbf{1}}\mathsf{G}$, with $[\cdot,\cdot]$ defined by

$$[\dot{g}(0), \dot{h}(0)] = \frac{\partial^2}{\partial t \partial s}\left(g(t)\,h(s)\,g(t)^{-1}\,h(s)^{-1}\right)\Big|_{t=s=0}$$
$$g(0) = h(0) = \mathbf{1}$$

  forms a Lie algebra $\mathfrak{g}$, also called the tangent algebra of the Lie group G.
- For any Lie group homomorphism $f : \mathsf{G} \to \mathsf{H}$ we have that $T_{\mathbf{1}}f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. That is we have a functor **LGrp** $\to$ **LAlg**.

The universal covering Lie group

- Given any Lie algebra $\mathfrak{g}$ there exists a simply connected Lie group $\widetilde{\mathsf{G}}$ with Lie algebra isomorphic to $\mathfrak{g}$.
- Any connected Lie group with Lie algebra $\mathfrak{g}$ is isomorphic to a quotient $\widetilde{\mathsf{G}}/\mathsf{N}$ where N is a discrete central subgroup.
- $\widetilde{\mathsf{G}}$ is called the universal covering group and is determined up to isomorphism.



Control systems

- A (smooth) dynamical system on a Lie group G is given by

$$\dot{g} = X(g), \; g \in \mathsf{G}$$

where $X : \mathsf{G} \to T\mathsf{G}$ is a (smooth) vector field on G.

- A control system is a family of dynamical systems

$$\dot{g} = X_u(g), \; g \in \mathsf{G}, u \in \mathbb{R}^\ell$$

with vector fields X_u parametrised by $u \in \mathbb{R}^\ell$.

- A absolutely continuous curve $g(t)$ is a trajectory of control system if it is a solution of the dynamic equation corresponding to some admissible control $u(t)$.
- A system is controllable if for any two points $g_0, g_1 \in \mathsf{G}$ there exists a trajectory taking g_0 to g_1 .

Left invariant systems

- A left invariant vector field X on a Lie group G is one which makes the following diagram commute for every $g \in \mathsf{G}$

$$\begin{tikzcd} \mathsf{G} \arrow[r, "L_g"] \arrow[d, "X"'] & \mathsf{G} \arrow[d, "X"] \\ T\mathsf{G} \arrow[r, "TL_g"'] & \mathsf{G} \end{tikzcd}$$

That is to say: $X(g) = TL_g \cdot X(\mathbf{1})$.

Definition

A left invariant control system is a control system where all the vector fields X_u are left invariant.

LiCAS systems

Definition

A left invariant control affine system Σ is a pair (G, Ξ) where

- The state space G is a finite dimensional real Lie group,
- The dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is of the form

$$\Xi : (g, u) \mapsto TL_g \cdot \left(A_0 + \sum_{i=1}^{\ell} u_i A_i \right)$$

where $\{A_i\}_{i=\overline{1,\ell}}$ is a linearly independent set.

- We define the trace Γ of a system Σ as $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$.
- If Σ is controllable then G is connected and $\operatorname{Lie}\Gamma = \mathfrak{g}$. Such systems are called proper systems.

Detached feedback equivalence

Definition

Two LiCAS systems Σ and Σ' are detached feedback equivalent if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that the diagram

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^\ell & \xrightarrow{\phi \times \varphi} & \mathsf{G}' \times \mathbb{R}^{\ell'} \\ {\scriptstyle \Xi}\downarrow & & \downarrow{\scriptstyle \Xi'} \\ T\mathsf{G} & \xrightarrow[T\phi]{} & T\mathsf{G}' \end{array}$$

commutes, that is $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$.

- If Σ and Σ' are d.f.e. then ϕ maps trajectories to trajectories and hence if one is controllable so is the other.

Detached feedback equivalence

Theorem

Two proper LiCAS systems $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ are detached feedback equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.

- Require that G and G' are isomorphic Lie groups, $\dim \Gamma = \dim \Gamma'$ and $0 \in \Gamma \Leftrightarrow 0 \in \Gamma'$.
- So for a fixed Lie group G two systems Σ and Σ' are equivalent iff

$$\begin{aligned} \Gamma \sim \Gamma' \; &\Leftrightarrow \exists \phi \in \mathrm{Aut}\,\mathsf{G}, \; T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma', \\ &\Leftrightarrow \exists \psi \in d\,\mathrm{Aut}\,\mathsf{G}, \; \psi \cdot \Gamma = \Gamma'. \end{aligned}$$

Classification problem

Problem statement

Classify all proper LiCAS objects with Lie algebra $\mathfrak{g}$ by detached feedback equivalence.

Approach:

- Separate into connected Lie groups with Lie algebra (isomorphic to) $\mathfrak{g}$;
- Separate into (ℓ, ε)-affine subspaces of $\mathfrak{g}$, i.e., with dimension ℓ and homogeneity ε ($\varepsilon = 0 \Leftrightarrow 0 \in \Gamma$);
- Classify such full rank affine subspaces by relation $d\,\mathrm{Aut}\,\mathsf{G}$.

The Oscillator group

- We start with simply connected Lie group with given algebra $\mathfrak{h}_3^\diamond$.
- Have matrix representation of $\widetilde{\mathrm{H}}_3^\diamond$; typical element $m(x,y,z,\theta)$ with $x,y,z,\theta \in \mathbb{R}$

Typical element of $\widetilde{\mathrm{H}}_3^\diamond$

$$m(x,y,z,\theta) = \begin{bmatrix} 1 & -x\cos\theta + y\sin\theta & x\sin\theta + y\cos\theta & -2z & 0 \\ 0 & \cos\theta & -\sin\theta & y & 0 \\ 0 & \sin\theta & \cos\theta & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^\theta \end{bmatrix}$$

The Oscillator group

- Lie subgroups $\mathrm{H}_3 = \{m(x,y,z,0) \mid x,y,z \in \mathbb{R}\}$ and $\widetilde{\mathrm{SO}}(2) = \{m(0,0,0,\theta) \mid \theta \in \mathbb{R}\}$ isomorphic to the Heisenberg group and $(\mathbb{R},+)$ respectively.
- $\widetilde{\mathrm{H}}_3^\diamond$ decomposes as a semidirect product of Lie subgroups H_3 and $\widetilde{\mathrm{SO}}(2)$; $\widetilde{H}_3^\diamond = \mathrm{H}_3 \rtimes \widetilde{\mathrm{SO}}(2)$.
- $\widetilde{\mathrm{H}}_3^\diamond$ is diffeomorphic to the direct product of H_3 and $\widetilde{\mathrm{SO}}(2)$. Thus $\widetilde{\mathrm{H}}_3^\diamond$ is diffeomorphic to $\mathbb{R}^4$.
- Any element $m(x,y,z,\theta) \in \tilde{H}_3^\diamond$ may be decomposed as

$$m(x,y,z,\theta) = m(x,y,z,0)\, m(0,0,0,\theta).$$

The Oscillator algebra

- The Lie algebra of $\widetilde{\mathrm{H}}_3^\diamond$ has typical element $M(x, y, z, \theta)$ with $x, y, z, \theta \in \mathbb{R}$.

Typical element of $\mathfrak{h}_3^\diamond$

$$M(x,y,z,\theta) = \begin{bmatrix} 0 & -x & y & -2z & 0 \\ 0 & 0 & -\theta & y & 0 \\ 0 & \theta & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta \end{bmatrix}$$

- Let $E_1 = M(1,0,0,0), \ldots, E_4 = M(0,0,0,1)$. Then $\{E_i\}_{i=\overline{1,4}}$ is basis for $\mathfrak{h}_3^\diamond$. Non-zero commutator relations:

$$[E_1, E_2] = E_3 \qquad [E_1, E_4] = E_2 \qquad [E_2, E_4] = -E_1$$

Discrete subgroups of $\widetilde{\mathrm{H}}_3^\diamond$

- $Z(\widetilde{\mathrm{H}}_3^\diamond) = \{m(0,0,z,\theta) \mid z \in \mathbb{R}, \theta \in 2\pi\mathbb{Z}\}$.

Proposition

There are three types of non trivial discreet central subgroups of $\widetilde{\mathrm{H}}_3^\diamond$ up to being related by an element of $\mathrm{Aut}\,\widetilde{\mathrm{H}}_3^\diamond$, namely

$$\begin{aligned} n\mathrm{N}_1 &= \{m(0,0,0,2n\pi\theta) \mid \theta \in \mathbb{Z}\}\,, \quad n \in \mathbb{N} \\ \mathrm{N}_2 &= \{m(0,0,z,0) \mid z \in \mathbb{Z}\} \\ n\mathrm{N}_1 \oplus \mathrm{N}_2 &= \{m(0,0,z,2n\pi\theta) \mid \theta, z \in \mathbb{Z}\}\,, \quad n \in \mathbb{N}. \end{aligned}$$

- The proof involves showing that any central discrete subgroup N is isomorphic to one of the given types by considering the discrete subgroups $\mathrm{N} \cap \widetilde{\mathrm{SO}}(2)$ and $\mathrm{N} \cap \mathrm{H}_3$ of H_3 and $\widetilde{\mathrm{SO}}(2)$ respectively.

Connected Lie groups with Lie algebra $\mathfrak{h}_3^\diamond$

- There are four types of connected Lie groups: one simply connected; three correspond to three types of discrete subgroups by quotients $\widetilde{\mathsf{H}}_3^\diamond / N$. We have following commutative diagram of covering Lie group morphisms.

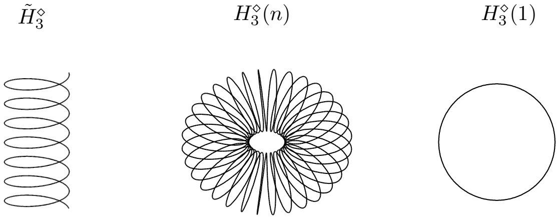
Structure of connected Lie groups with Lie algebra $\mathfrak{h}_3^\diamond$

$$\begin{tikzcd}
\widetilde{\mathsf{H}}_3^\diamond \arrow[r] \arrow[d] & \mathsf{H}_3^\diamond(n) = \widetilde{\mathsf{H}}_3^\diamond / n\mathsf{N}_1 \arrow[r] \arrow[d] & \mathsf{H}_3^\diamond = \widetilde{\mathsf{H}}_3^\diamond / \mathsf{N}_1 \arrow[d] \\
\widetilde{\mathsf{H}}_3^\diamond / \mathsf{N}_2 \arrow[r] & \widetilde{\mathsf{H}}_3^\diamond / (n\mathsf{N}_1 \oplus \mathsf{N}_2) \arrow[r] & \widetilde{\mathsf{H}}_3^\diamond / (\mathsf{N}_1 \oplus \mathsf{N}_2)
\end{tikzcd}$$

- $\mathsf{H}_3^\diamond(n) = \mathsf{H}_3 \rtimes \mathsf{SO}_n(2)$; $\mathsf{SO}_n(2)$ is n-fold cover of $\mathsf{SO}\,(2)$.

Connected Lie groups with Lie algebra $\mathfrak{h}_3^\diamond$

- $\widetilde{\mathsf{H}}_3^\diamond$, $\mathsf{H}_3^\diamond(n) = \widetilde{\mathsf{H}}_3^\diamond / n\mathsf{N}_1$ have linear representation; $\widetilde{\mathsf{H}}_3^\diamond / \mathsf{N}_2$, $\widetilde{\mathsf{H}}_3^\diamond / (n\mathsf{N}_1 \oplus \mathsf{N}_2)$ have no linear representation.
- We will look at $\widetilde{\mathsf{H}}_3^\diamond$ and $\mathsf{H}_3^\diamond(n)$.
- Topologically we have that $\widetilde{\mathsf{H}}_3^\diamond \sim \mathbb{R}^3 \times \mathbb{R}$ and $\mathsf{H}_3^\diamond(n) \sim \mathbb{R}^3 \times \mathbb{T}$. We illustrate the situation.



Automorphisms of $\widetilde{\mathsf{H}}_3^\diamond$

- Universal cover $\widetilde{\mathsf{G}}$: $d\,\mathrm{Aut}\,\widetilde{\mathsf{G}} = \mathrm{Aut}\,\mathfrak{g}$.
- Seek linear map $\psi : \psi \cdot [E_i, E_j] = [\psi \cdot E_i, \psi \cdot E_j],\; i,j = \overline{1,4}$.

Proposition

$$\mathrm{Aut}\,\mathfrak{h}_3^\diamond = \left\{ \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & w \\ 0 & 0 & 0 & k \end{bmatrix} \in \mathrm{GL}(4,\mathbb{R}) \;\middle|\; x,y,u,v,w \in \mathbb{R},\; k \in \{-1,1\} \right\}$$

Automorphisms of $\mathsf{H}_3^\diamond(n)$

- Connected Lie group G : $d\,\mathrm{Aut}\,\mathsf{G} \leq \mathrm{Aut}\,\mathfrak{g}$.
- Seek elements of $\mathrm{Aut}\,\mathfrak{h}_3^\diamond$ that lift to automorphisms of group.

Proposition

$$d\,\mathrm{Aut}\,\mathsf{H}_3^\diamond(n) = \left\{ \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & \frac{1}{2}k(u^2+v^2) \\ 0 & 0 & 0 & k \end{bmatrix} \in \mathsf{GL}(4,\mathbb{R}) \;\middle|\; x,y,u,v,w \in \mathbb{R},\ k \in \{-1,1\} \right\}$$

Classification of $\widetilde{\mathrm{H}}_3^\diamond$

Proposition

Any $(2,0)$-trace (i.e., $0 \in \Gamma$, $\dim \Gamma = 2$) is related to $\langle E_1, E_4 \rangle$.

Proof sketch. Arbitrary trace Γ of given type:

$\Gamma = \left\langle \sum_{i=1}^4 a_i E_i, \sum_{i=1}^4 b_i E_i \right\rangle$.

Full rank implies $a_4 \neq 0$ or $b_4 \neq 0$.

$\Gamma = \left\langle \sum_{i=1}^3 a_i' E_i + E_4, \sum_{i=1}^4 b_i' E_i \right\rangle$.

$$\psi = \begin{bmatrix} 1 & 0 & 0 & -a_1' \\ 0 & 1 & 0 & -a_2 \\ -a_1 & -a_2 & 1 & -a_3 + a_1^2 + a_2^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\psi \cdot \Gamma = \left\langle E_4, \sum_{i=1}^3 b_i'' E_i \right\rangle$.

...

$\Gamma \sim \langle E_1, E_4 \rangle$.

Classification of $\mathsf{H}_3^\diamond(n)$

Proposition

Any $(2,0)$-trace (i.e., $0 \in \Gamma$, $\dim \Gamma = 2$) is related to exactly one of

$$\begin{aligned}
\Gamma_1 &= \langle E_1, E_3 + E_4 \rangle \\
\Gamma_2 &= \langle E_1, E_4 \rangle \\
\Gamma_3 &= \langle E_1, -E_3 + E_4 \rangle
\end{aligned}$$

Proof sketch. We show $\Gamma_1 \not\sim \Gamma_2$. Suppose $\exists \psi$, $\psi \cdot \Gamma_2 = \Gamma_1$

$$\psi = \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & \frac{1}{2}k(u^2+v^2) \\ 0 & 0 & 0 & k \end{bmatrix}.$$

Leads to contradiction.

Conclusion

- Using the same approach we have classified all LiCAS systems with algebra $\mathfrak{h}_3^\diamond$.
- Using classification we have been able to characterise controllable systems:
  - $\widetilde{\mathsf{H}}_3^\diamond$: controllable iff full rank and projection of Γ^0 onto E_4 not $\{0\}$;
  - $\mathsf{H}_3^\diamond(n)$: controllable iff full rank.
- Classification of $\widetilde{\mathsf{H}}_3^\diamond$ systems also provides local classification.

- Outlook
  - Optimal control problem on $\mathsf{H}_3^\diamond$ systems.