Control Affine Systems on 3D Lie Groups

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Outline

1. Introduction
2. Systems: equivalence and controllability
3. Conclusion

Outline

1 Introduction

2 Systems: equivalence and controllability

3 Conclusion

Problem statement

Study equivalence and controllability of control systems

Systems

- left-invariant control affine
- $\mathsf{SE}(2)$, $\mathsf{SE}(1,1)$, $\mathsf{SO}(3)$, $\mathsf{SO}(2,1)_0$

Equivalence

- detached feedback equivalence

Controllability

- equivalence class
- characterize

Left-invariant control affine systems

System Σ

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

$A + \langle B_1, \ldots, B_\ell \rangle$ — ℓ-dim affine subspace of $\mathfrak{g}$

Trajectory $g(\cdot) : [0, T] \to \mathsf{G}$

- for admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- integral curve of time-varying vector field

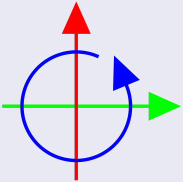
Controllable

- Exists a trajectory between any two points $g_0, g_1 \in \mathsf{G}$
- G connected, Σ has full-rank — necessary

Classical 3D groups

Euclidean group SE(2)

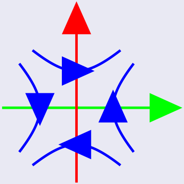
$$\begin{bmatrix} 1 & 0 & 0 \\ x & \cos z & -\sin z \\ y & \sin z & \cos z \end{bmatrix}$$



$$[E_2, E_3] = E_1$$
$$[E_3, E_1] = E_2$$
$$[E_1, E_2] = 0$$

Semi-Euclidean group SE(1, 1)

$$\begin{bmatrix} 1 & 0 & 0 \\ x & \cosh z & -\sinh z \\ y & -\sinh z & \cosh z \end{bmatrix}$$

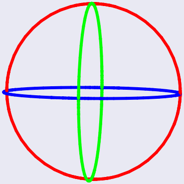


$$[E_2, E_3] = E_1$$
$$[E_3, E_1] = -E_2$$
$$[E_1, E_2] = 0$$

Classical 3D groups

Orthogonal group $\mathsf{SO}(3)$

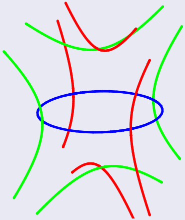
$$g^\top g = \mathbf{1}$$
$$\det g = 1$$



$$[E_2, E_3] = E_1$$
$$[E_3, E_1] = E_2$$
$$[E_1, E_2] = E_3$$

Pseudo-orthogonal group $\mathsf{SO}(2,1)$

$$g^\top J g = J$$
$$J = \mathrm{diag}(1, 1, -1)$$
$$\det g = 1$$



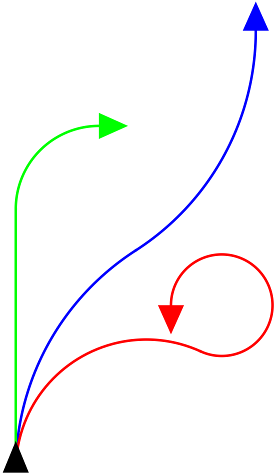
$$[E_2, E_3] = E_1$$
$$[E_3, E_1] = E_2$$
$$[E_1, E_2] = -E_3$$

$$\Sigma : \; \dot{g} = g(E_2 + uE_3)$$

$$\dot{g} = \begin{bmatrix} 1 & 0 & 0 \\ x & \cos z & -\sin z \\ y & \sin z & \cos z \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -u \\ 1 & u & 0 \end{bmatrix}$$

Parametrically

$$\dot{x} = -\sin z \qquad x(0) = 0$$
$$\dot{y} = \cos z \qquad y(0) = 0$$
$$\dot{z} = u \qquad z(0) = 0$$

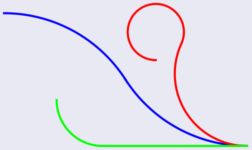


Equivalence

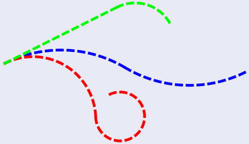
$$\Sigma : \; \dot{g} = g(A + u_1 B_1 + \cdots + u_\ell B_\ell)$$
$$\Sigma' : \; \dot{g} = g(A' + u_1 B_1' + \cdots + u_\ell B_\ell')$$

are equivalent if $\exists\, \phi \in \mathrm{Aut}\,(\mathrm{G})$ relating trajectories

Example



$\Sigma : \; E_3 - E_1 + uE_3$



$\Sigma' : \; \frac{1}{4}(2E_1 + E_2) + uE_3$

Characterization

$$\exists\, \psi \in d\,\mathrm{Aut}\,(\mathrm{G}), \qquad \psi \cdot (A + \langle B_1, \ldots, B_\ell \rangle) = A' + \langle B_1', \ldots, B_\ell' \rangle$$

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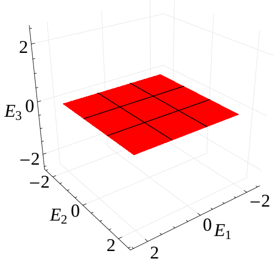
Euclidean group SE (2)

$d\,\mathrm{Aut}\,(\mathrm{SE}\,(2))$

$$\left\{ \begin{bmatrix} x & y & v \\ -\sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix} : x, y, z, v, w \in \mathbb{R},\ \sigma = \pm 1,\ x^2 + y^2 \neq 0 \right\}$$

- preserves $\langle E_1, E_2 \rangle$
- transitive on $\langle E_1, E_2 \rangle \setminus \{\mathbf{0}\}$

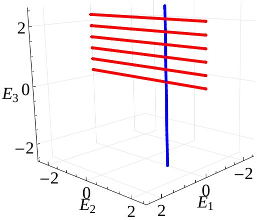


Euclidean group SE (2)

Single-input

$$\Sigma_1^{(1)} : \; E_2 + uE_3$$

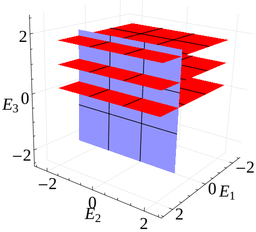
$$\Sigma_{2,\alpha}^{(1)} : \; \alpha E_3 + uE_2$$



Two-input

$$\Sigma_1^{(2)} : \; E_1 + u_1E_2 + u_2E_3$$

$$\Sigma_{2,\alpha}^{(2)} : \; \alpha E_3 + u_1E_1 + u_2E_2$$



Controllability

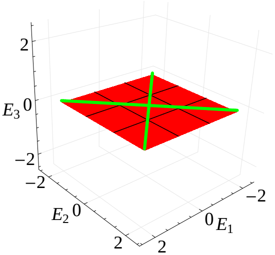
All full-rank systems are controllable [Bonnard, Jurdjevic, et al, 1982]

Semi-Euclidean group $\mathsf{SE}(1,1)$

$d\,\mathsf{Aut}(\mathsf{SE}(1,1))$

$$\left\{ \begin{bmatrix} x & y & v \\ \sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix} : x, y, z, v, w \in \mathbb{R},\ \sigma = \pm 1,\ x^2 - y^2 \neq 0 \right\}$$

- preserves subsets
  - $\langle E_1, E_2 \rangle$
  - $\mathcal{C} = \langle E_1 + E_2 \rangle \cup \langle E_1 - E_2 \rangle$
- transitive on
  - $\langle E_1, E_2 \rangle \setminus \mathcal{C}$
  - $\mathcal{C} \setminus \{\mathbf{0}\}$

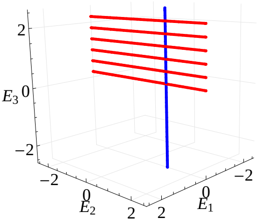


Semi-Euclidean group $SE(1,1)$

Single-input

$$\Sigma_1^{(1)} : \; E_1 + uE_3$$

$$\Sigma_{2,\alpha}^{(1)} : \; \alpha E_3 + uE_2$$

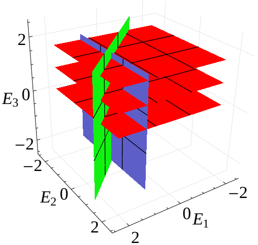


Two-input

$$^*\Sigma_1^{(2)} : \; E_1 + u_1 E_2 + u_2 E_3$$

$$\Sigma_2^{(2)} : \; E_1 + u_1(E_1 + E_2) + u_2 E_3$$

$$\Sigma_{3,\alpha}^{(2)} : \; \alpha E_3 + u_1 E_1 + u_2 E_2$$



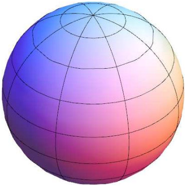
Controllability (simply-connected, completely solvable)

Controllable $\iff$ $B_1, \ldots, B_\ell$ generate $\mathfrak{se}(1,1)$ [Sachkov, 2009]

Orthogonal group $\mathsf{SO}(3)$

$$d\,\mathsf{Aut}(\mathsf{SO}(3)) = \mathsf{SO}(3)$$

- preserves $A \bullet B = a_1 b_1 + a_2 b_2 + a_3 b_3$
- preserves spheres $\mathcal{S}_\alpha = \{A : A \bullet A = \alpha > 0\}$
- transitive on spheres $\mathcal{S}_\alpha$

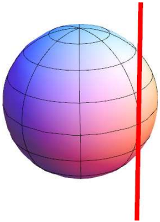


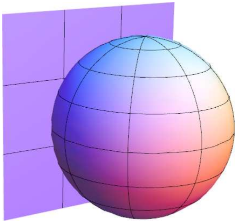
Orthogonal group SO (3)

Systems

$$\Sigma_\alpha^{(1)}: \ \alpha E_2 + u E_3$$

$$\Sigma_\alpha^{(2)}: \ \alpha E_1 + u_1 E_2 + u_2 E_3$$





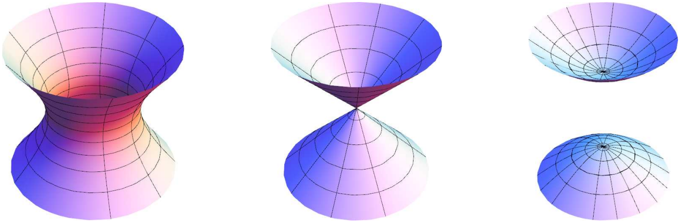
Controllability

Compact $\Rightarrow$ all full-rank systems are controllable

Pseudo-orthogonal group $\mathrm{SO}(2,1)_0$

$$d\,\mathrm{Aut}(\mathrm{SO}(2,1)_0) = \mathrm{SO}(2,1)$$

- preserves $A \odot B = a_1 b_1 + a_2 b_2 - a_3 b_3$
- preserves hyperboloids $\mathcal{H}_\alpha = \{A : A \odot A = \alpha,\, A \neq 0\}$
- transitive on hyperboloids $\mathcal{H}_\alpha$



Pseudo-orthogonal group $SO(2,1)_0$

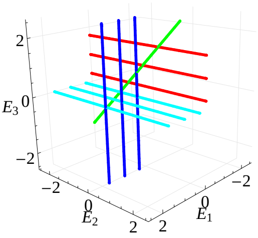
Single-input

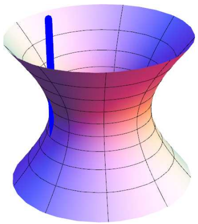
$^*\Sigma_1^{(1)}: \; E_3 + u(E_2 + E_3)$

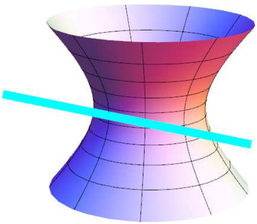
$^*\Sigma_{2,\alpha}^{(1)}: \; \alpha E_1 + u E_3$

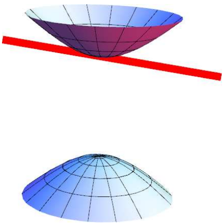
$^*\Sigma_{3,\alpha}^{(1)}: \; \alpha E_3 + u E_2$

$\Sigma_{4,\alpha}^{(1)}: \; \alpha E_1 + u E_2$







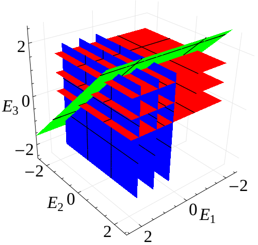


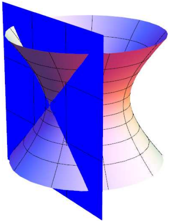
Two-input

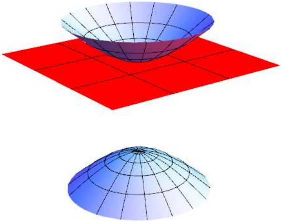
$^*\Sigma_1^{(2)}$: $E_3 + u_1 E_1 + u_2 (E_2 + E_3)$

$^*\Sigma_2^{(2)}$: $\alpha E_1 + u_1 E_2 + u_2 E_3$

$^*\Sigma_{3,\alpha}^{(2)}$: $\alpha E_3 + u_1 E_1 + u_2 E_2$







Pseudo-orthogonal group $\mathrm{SO}(2,1)_0$

Controllability

A full-rank system is controllable

$$\iff \quad \exists C \in A + \langle B_1, \ldots, B_\ell \rangle \ \text{ such that } \ C \odot C < 0$$

- Suppose $\exists C$ such that $C \odot C < 0$
  - $t \to \exp(tC)$ is periodic
  - controllable [Jurdjevic and Sussmann, 1972]
- If no such C exists
  - system equivalent to $\Sigma^{(1)}_{3,\alpha}$, not controllable

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1 Introduction

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Conclusion

Summary

- Classified systems on 3D matrix Lie groups
- Characterized controllability

Outlook

- Organize systematically results
- Optimal control (and classification)