Quadratic Hamilton-Poisson Systems on $\mathfrak{se}(1, 1)^*$
Equivalence, Stability and Integrability

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Outline

1. Hamilton-Poisson formalism
2. The semi-Euclidean Lie algebra
3. Classification of systems
4. Stability analysis
5. Integration
Introduction

Context

Study a class of Hamilton-Poisson systems relating to optimal control problems on Lie groups.

Objects

- quadratic Hamilton-Poisson systems on duals of Lie algebras

Equivalence

- equivalence under affine isomorphisms

Problem

- classify Hamilton-Poisson systems under affine equivalence
- investigate stability nature of equilibria
- find integral curves of systems.
# Lie-Poisson structures

**Poisson bracket \(\{\cdot, \cdot\} \) on \(\mathfrak{g}^*\)**

A skew-symmetric, bilinear map \(C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \rightarrow C^\infty(\mathfrak{g}^*)\) satisfying:

- **Jacobi identity**
- \(\{\cdot, F\}\) is a derivation, \(\forall F \in C^\infty(\mathfrak{g}^*)\).

(Minus) **Lie-Poisson space** \(\mathfrak{g}^- = (\mathfrak{g}^*, \{\cdot, \cdot\})\)

\[
\{F, G\}(p) = -p([dF(p), dG(p)]).
\]

**Linear Poisson) automorphisms**

Linear isomorphisms \(\Psi : \mathfrak{g}^* \rightarrow \mathfrak{g}^*\) that preserve the Poisson bracket:

\[
\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}, \quad \forall F, G \in C^\infty(\mathfrak{g}^*).
\]
Hamiltonian formalism

Hamiltonian vector fields

For every Hamiltonian function $H \in C^\infty(g^*)$ there is a unique vector field $\vec{H} \in \text{Vec}(g^*)$ such that

$$\vec{H}[F] = \{F, H\}, \quad \forall F \in C^\infty(g^*).$$

Equations of motion

A curve $p(\cdot)$ is an integral curve of $\vec{H}$ if

$$\frac{d}{dt} p(t) = \vec{H}(p(t)).$$

In coordinates,

$$\frac{d}{dt} p_i(t) = -p([E_i, dH(p)]).$$
**Conservation of energy**

If \( p(\cdot) \) is an integral curve of \( \vec{H} \), then \( H(p(t)) \) is constant in \( t \).

**Casimir functions**

Functions \( C \in C^\infty(g^*) \) that Poisson commute with every other function:

\[
\{ C, F \} = 0, \quad \forall F \in C^\infty(g^*). 
\]

Integral curves of \( \vec{H} \) evolve on the intersection of the surfaces

\[
H(p) = \text{const.} \quad \text{and} \quad C(p) = \text{const.}
\]
Stability of equilibria

**Equilibria**

An equilibrium point of $\vec{H}$ is a point $p_e \in \mathfrak{g}^*$ such that $\vec{H}(p_e) = 0$.

**Lyapunov stability nature of $p_e$**

- (Lyapunov) stable if for every neighbourhood $N$ of $p_e$ there exists a neighbourhood $N' \subseteq N$ of $p_e$ such that, for every integral curve $p(\cdot)$ of $\vec{H}$ with $p(0) \in N'$, we have $p(t) \in N$ for all $t > 0$.
- (Lyapunov) unstable if it is not stable.
Lyapunov stability

(a) Stability

(b) Instability
Lyapunov stability

Energy-Casimir method

Suppose there exist
- constants of motion $C_1, \ldots, C_k$ (i.e. $\{C_i, H\} = 0$)
- $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$

such that
- $d(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(p_e) = 0$
- $d^2(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(p_e)|_{W \times W}$ is positive definite, where

$$W = \ker dH(p_e) \cap \ker dC_1(p_e) \cap \cdots \cap \ker dC_k(p_e).$$

Then $p_e$ is (Lyapunov) stable.
Spectral stability

Spectral stability nature of $p_e$

- **spectrally stable** if all eigenvalues of $\mathbf{D} \vec{H}(p_e)$ have non-positive real parts.
- **spectrally unstable** if it is not spectrally stable.

Lyapunov stability $\Rightarrow$ Spectral stability
Quadratic Hamilton-Poisson systems

Quadratic HP systems \((\mathfrak{g}^*, H_{A,Q})\)

The Hamiltonian \(H_{A,Q}\) is given by

\[
H_{A,Q}(p) = L_A(p) + H_Q(p) = p(A) + Q(p) \quad (A \in \mathfrak{g}).
\]

- \(Q\) is a **quadratic form** on \(\mathfrak{g}^*\)
- in coordinates: \(H_{A,Q}(p) = pA + \frac{1}{2} pQp^\top\)
- \(H_{A,Q}\) is **homogeneous** if \(A = 0\); otherwise, **inhomogeneous**.

**Restriction**

- \(Q\) is **positive semidefinite**.
Equivalence of systems

Affine equivalence ($A$-equivalence)

$H_{A,Q}$ and $H_{B,R}$ are $A$-equivalent if there exists an affine isomorphism $\Psi : g^* \to g^*$, $p \mapsto \Psi_0(p) + q$ such that

$$\Psi_0 \cdot \vec{H}_{A,Q} = \vec{H}_{B,R} \circ \Psi.$$  

We write $H_{A,Q} \sim H_{B,R}$.

Sufficient conditions

$H_{A,Q}$ is $A$-equivalent to

- $H_{A,Q} \circ \Psi$, where $\Psi : g^* \to g^*$ is a linear Poisson automorphism
- $H_{A,Q} + C$, where $C$ is a Casimir function
- $H_{A,rQ}$, where $r \neq 0$.  

The semi-Euclidean Lie algebra

$\mathfrak{se}(1, 1)$

$\mathfrak{se}(1, 1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$

Standard basis

$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Commutators

$[E_2, E_3] = -E_1$

$[E_3, E_1] = E_2$

$[E_1, E_2] = 0$
Hamilton-Poisson systems on $\mathfrak{se}(1, 1)^*$

Dual basis $(E_1^*, E_2^*, E_3^*)$

\[
E_i^*(E_j) = \delta_{ij}, \quad 1 \leq i, j \leq 3.
\]

Equations of motion

\[
\begin{align*}
\dot{p}_1 &= \frac{\partial H}{\partial p_3} p_2 \\
\dot{p}_2 &= \frac{\partial H}{\partial p_3} p_1 \\
\dot{p}_3 &= -\frac{\partial H}{\partial p_1} p_2 - \frac{\partial H}{\partial p_2} p_1
\end{align*}
\]

Casimir function

The function $C : (p_1, p_2, p_3) \mapsto p_1^2 - p_2^2$ is a Casimir on $\mathfrak{se}(1, 1)^*$. 
Classification: the homogeneous case

**Proposition**

Any HP system \((\mathfrak{se}(1,1)^*, H_Q)\) is A-equivalent to exactly one of the following systems:

\[
\begin{align*}
H_0(p) &= 0 \\
H_1(p) &= \frac{1}{2} p_1^2 \\
H_2(p) &= \frac{1}{2} (p_1 + p_2)^2 \\
H_3(p) &= \frac{1}{2} p_3^2 \\
H_4(p) &= \frac{1}{2} (p_1^2 + p_3^2) \\
H_5(p) &= \frac{1}{2} [(p_1 + p_2)^2 + p_3^2]
\end{align*}
\]

**Method of proof**

- simplify representatives using sufficient conditions for A-equivalence
- result: collection of potential representatives
- confirm that representatives are not equivalent.
Classification of inhomogeneous systems

Approach (for each $i = 0, \ldots, 5$)

- assume $H_{A,Q} = L_A + H_i$
- simplify $L_A$ using automorphisms that leave $H_i$ invariant
- employ affine isomorphisms for further simplification
- verify that representatives are not equivalent.
Example: systems associated to $H_3(p) = \frac{1}{2} p_3^2$

**Lemma**

There exists an automorphism $\Psi$ such that $H_3 \circ \Psi = H_3$ and $L_A \circ \Psi$ is exactly one of $L_{E_1 + \beta E_3}$, $L_{E_1 + E_2 + \gamma E_3}$ or $L_{\alpha E_3}$, where $\alpha > 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$.

- From the lemma, $L_A + H_3$ is $A$-equivalent to one of
  
  $G_{1,\beta}(p) = p_1 + \beta p_3 + \frac{1}{2} p_3^2$
  
  $G_{2,\gamma}(p) = p_1 + p_2 + \gamma p_3 + \frac{1}{2} p_3^2$
  
  $G_{3,\alpha}(p) = \alpha p_3 + \frac{1}{2} p_3^2$

- using $\Psi: p \mapsto p + \beta E_3^*$, we have $G_{1,\beta} \sim G_{1,0}$

- similarly, $G_{2,\gamma} \sim G_{2,0}$ and $G_{3,\alpha} \sim G_{3,0}$

- verify that $G_{1,0}$, $G_{2,0}$ and $G_{3,0}$ are not equivalent.
Example: systems associated to \( H_3(p) = \frac{1}{2}p_3^2 \)

**Proposition**

Any HP system \((s \mathcal{C}(1, 1)^*, H_A, Q)\) of the form \(H_A, Q = L_A + H_3\) is \(A\)-equivalent to exactly one of the following systems:

\[
\begin{align*}
H_1^{(3)}(p) &= p_1 + \frac{1}{2}p_3^2 \\
H_2^{(3)}(p) &= p_1 + p_2 + \frac{1}{2}p_3^2 \\
H_3^{(3)}(p) &= \frac{1}{2}p_3^2.
\end{align*}
\]
Classification of inhomogeneous systems \( (\mathfrak{se}(1,1)^*, H_{A,Q}) \)

\[
H_1^{(0)}(p) = p_1 \\
H_2^{(0)}(p) = \alpha p_3
\]

\[
H_1^{(1)}(p) = p_1 + \frac{1}{2} p_1^2 \\
H_2^{(1)}(p) = p_1 + p_2 + \frac{1}{2} p_1^2 \\
H_3^{(1)}(p) = \alpha p_3 + \frac{1}{2} p_1^2
\]

\[
H_1^{(2)}(p) = p_1 + \frac{1}{2} (p_1 + p_2)^2 \\
H_2^{(2)}(p) = p_1 + p_2 + \frac{1}{2} (p_1 + p_2)^2 \\
H_3^{(2)}(p) = \delta p_3 + \frac{1}{2} (p_1 + p_2)^2
\]

\[
H_1^{(3)}(p) = p_1 + \frac{1}{2} p_3^2 \\
H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2} p_3^2 \\
H_3^{(3)}(p) = \frac{1}{2} p_3^2
\]

\[
H_1^{(4)}(p) = \alpha p_1 + \frac{1}{2} (p_1^2 + p_3^2) \\
H_2^{(4)}_{\alpha_1,\alpha_2}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2} (p_1^2 + p_3^2)
\]

\[
H_1^{(5)}(p) = \alpha p_1 + \frac{1}{2} [(p_1 + p_2)^2 + p_3^2] \\
H_2^{(5)}(p) = p_1 - p_2 + \frac{1}{2} [(p_1 + p_2)^2 + p_3^2] \\
H_3^{(5)}(p) = \alpha (p_1 + p_2) + \frac{1}{2} [(p_1 + p_2)^2 + p_3^2]
\]

\[\alpha > 0, \, \alpha_1 > \alpha_2 > 0, \, \delta \neq 0\]
Stability analysis of $H_1^{(3)}(p) = p_1 + \frac{1}{2} p_2^2$

Equations of motion

\[
\begin{cases}
\dot{p}_1 = p_2 p_3 \\
\dot{p}_2 = p_1 p_3 \\
\dot{p}_3 = -p_2
\end{cases}
\]

Equilibria

\[
e_1^\mu = (\mu, 0, 0) \\
e_2^\nu = (0, 0, \nu)
\]

The states $e_2^\nu$ are unstable

- the linearisation is

\[
D \vec{H}_1^{(3)}(p) = \begin{bmatrix}
0 & p_3 & p_2 \\
p_3 & 0 & p_1 \\
-1 & -1 & 0
\end{bmatrix} \\
\Rightarrow 
D \vec{H}_1^{(3)}(e_2^\nu) = \begin{bmatrix}
0 & \nu & 0 \\
\nu & 0 & 0 \\
-1 & -1 & 0
\end{bmatrix}
\]

- the eigenvalues of $D \vec{H}_1^{(3)}(e_2^\nu)$ are $\lambda_1 = 0$, $\lambda_{2,3} = \pm \nu$

- therefore $e_2^\nu$ is (spectrally) unstable.
Stability analysis of $H_1^{(3)}(p) = p_1 + \frac{1}{2} p_3^2$

The states $e_1^\mu = (\mu, 0, 0)$, $\mu \leq 0$ are unstable

- consider the case $\mu = 0$ ($\mu < 0$ is similar)
- the curve

$$p(\cdot) : (−\infty, 0) \to se(1, 1)^*, \quad t \mapsto \left(−\frac{2}{t^2}, \frac{2}{t^2}, \frac{2}{t}\right)$$

is an integral curve of $\vec{H}_1^{(3)}$, with

$$\lim_{t \to -\infty} \|p(t) - e_1^0\| = 0.$$  

- thus for every neighbourhood $N$ of $e_1^0$, there exists $t_1 < 0$ such that $p(t_1) \in N$
- since $\lim_{t \to 0} \|p(t) - e_1^0\| = \infty$, the state $e_1^0$ is unstable.
Stability analysis of $H_1^{(3)}(p) = p_1 + \frac{1}{2} p_3^2$

The states $e^{\mu}_1 = (\mu, 0, 0), \mu > 0$ are stable

- Let $H_\lambda = \lambda_0 H_1^{(3)} + \lambda_1 C$, where $\lambda_0 = \mu, \lambda_1 = -\frac{1}{2}$
- Then $dH_\lambda(e^{\mu}_1) = 0$ and
  
  $$d^2 H_\lambda(e^{\mu}_1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

- Since $W = \ker dH_1^{(3)}(e^{\mu}_1) \cap \ker dC(e^{\mu}_1) = \text{span} \{ E_2^*, E_3^* \}$, the restriction $d^2 H_\lambda(e^{\mu}_1)|_{W \times W} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$ is positive definite
- Therefore the states $e^{\mu}_1, \mu > 0$ are stable.
Integration of $H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2}p_3^2$

Equations of motion

\[
\begin{align*}
\dot{p}_1 &= p_2p_3 \\
\dot{p}_2 &= p_1p_3 \\
\dot{p}_3 &= -(p_1 + p_2)
\end{align*}
\]

Sketch of integration

- let $\bar{p}(\cdot)$ be an integral curve of $H_2^{(3)}$
- let $h_0 = H_2^{(3)}(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$
- consider the case $c_0 > 0, h_0 < 0$

Figure: $c_0 > 0, h_0 < 0$
Case $c_0 > 0$, $h_0 < 0$

Sketch of integration, cont’d

- from $\dot{p}_3 = -(p_1 + p_2)$ and $h_0 = p_1(t) + p_2(t) + \frac{1}{2}p_3(t)^2$, we get the ODE
  $$\frac{d}{dt}p_3(t) = \frac{1}{2}p_3(t)^2 - h_0 \Rightarrow p_3(t) = 2\Omega \tan(\Omega t), \quad \Omega = \sqrt{-\frac{h_0}{2}}$$

- differentiate $p_3(t)$ to get
  $$p_1(t) + p_2(t) = 2\Omega^2 \sec^2(\Omega t)$$

- since $p_1(t)^2 - p_2(t)^2 = c_0$, we have
  $$p_1(t) - p_2(t) = \frac{c_0}{2\Omega^2} \cos^2(\Omega t)$$
Case \( c_0 > 0, \ h_0 < 0 \)

Sketch of integration, cont’d

- now solve the equation

\[
\begin{bmatrix}
1 & 1 \\
1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
\bar{p}_1(t) \\
\bar{p}_2(t) \\
\end{bmatrix}
= 
\begin{bmatrix}
2\Omega^2 \sec^2(\Omega t) \\
\frac{c_0}{2\Omega^2} \cos^2(\Omega t) \\
\end{bmatrix}
\]

- thus we have a **prospective** integral curve \( \bar{p}(\cdot) \)

- confirm that \( \dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t)) \)

- we can now make a statement regarding all integral curves of \( \vec{H}_2^{(3)} \) when \( c_0 > 0, \ h_0 < 0 \).
Integral curves of $\vec{H}_2^{(3)}$ with $c_0 > 0$, $h_0 < 0$

**Proposition**

Let $p(\cdot) : (-\varepsilon, \varepsilon) \rightarrow se(1, 1)^*$ be an integral curve of $\vec{H}_3^{(2)}$ such that $H_2^{(3)}(p(0)) = h_0 < 0$ and $C(p(0)) = c_0 > 0$.

(i) There exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$, where $\bar{p}(\cdot) : (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}) \rightarrow se(1, 1)^*$ is defined by

$$
\begin{align*}
\bar{p}_1(t) &= -\frac{1}{4\Omega^2} \left[ 4\Omega^4 \sec^2(\Omega t) + c_0 \cos^2(\Omega t) \right] \\
\bar{p}_2(t) &= -\frac{1}{4\Omega^2} \left[ 4\Omega^4 \sec^2(\Omega t) - c_0 \cos^2(\Omega t) \right] \\
\bar{p}_3(t) &= 2\Omega \tan(\Omega t).
\end{align*}
$$

Here $\Omega = \sqrt{-h_0/2}$.

(ii) $t \mapsto \bar{p}(t + t_0)$ is the unique maximal integral curve starting at $\bar{p}(t_0)$. 
Integral curves of $\vec{H}_2^{(3)}$ with $c_0 > 0$, $h_0 < 0$

Proof sketch

Item (i):
- show that $\exists \, t_0$ such that $\bar{p}(t_0) = p(0)$
- then $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem
- hence $p(t) = \bar{p}(t + t_0)$.

Item (ii):
- suppose $\exists$ an integral curve $q(\cdot) : (-\varepsilon', \varepsilon') \to se(1, 1)^*$ with $q(0) = \bar{p}(t_0)$ and $\frac{\pi}{2\Omega} \leq \varepsilon'$
- show that $\varepsilon' = \frac{\pi}{2\Omega}$
- uniqueness now follows from maximality of $t \mapsto \bar{p}(t + t_0)$. 
Conclusion

Further work on $\mathfrak{se}(1,1)^*$

- investigate remaining systems: $H_{1,\alpha}^{(4)}$, $H_{2,\alpha_1,\alpha_2}^{(4)}$, $H_{1,\alpha}^{(5)}$, $H_{2,\alpha}^{(5)}$ and $H_{3,\alpha}^{(5)}$
- link with optimal control problems

Further work on quadratic Hamilton-Poisson systems

- classify systems on all 3D Lie-Poisson spaces
- completed for the homogeneous case