Invariant Sub-Riemannian Structures on Lie Groups

Geodesics and Isometries

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Outline

Riemannian manifold: Euclidean space $\mathbb{E}^3$

Euclidean space $\mathbb{E}^3$

Metric tensor $\mathbf{g}$:

- for each $x \in \mathbb{R}^3$, we have inner product $\mathbf{g}_x$
- $\mathbf{g}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (dot product)

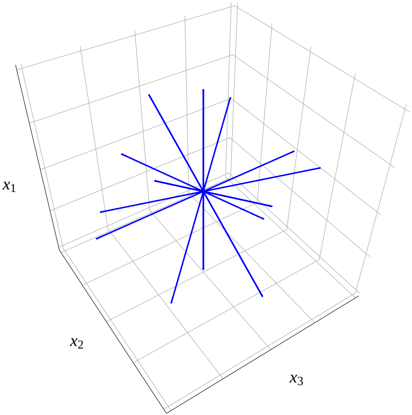
Length of a curve $\gamma(\cdot)$:

$$\ell(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}(\dot\gamma(t), \dot\gamma(t))}\, dt$$

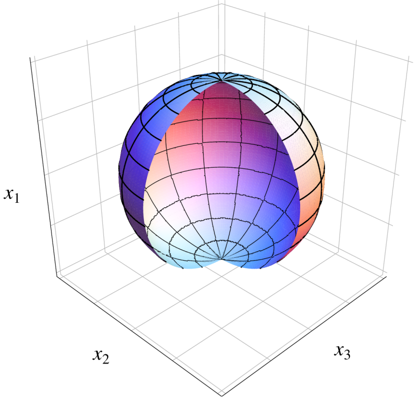
Distance:

$$d(x, y) = \inf\{\ell(\gamma(\cdot)) \,:\, \gamma(0) = x,\ \gamma(T) = y\}$$

Riemannian manifold: Euclidean space $\mathbb{E}^3$



geodesics: straight lines



$\mathcal{S}_1 = \{x \,:\, d(0,x) = 1\}$

Riemannian manifold: Heisenberg group

Invariant Riemannian structure on Heisenberg group

Metric tensor $\mathbf{g}$:

- for each $x \in \mathbb{R}^3$, we have inner product $\mathbf{g}_x$
- $\mathbf{g}_x = \begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & 0 \\ -x_2 & 0 & 1 + x_2^2 \end{bmatrix}$

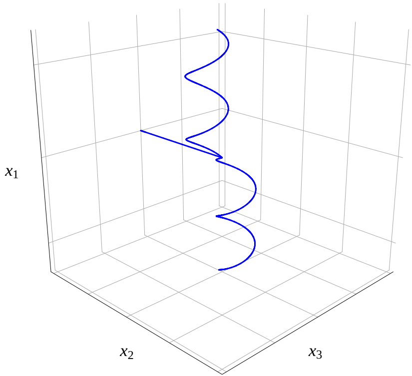
Length of a curve $\gamma(\cdot)$:

$$\ell(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt$$

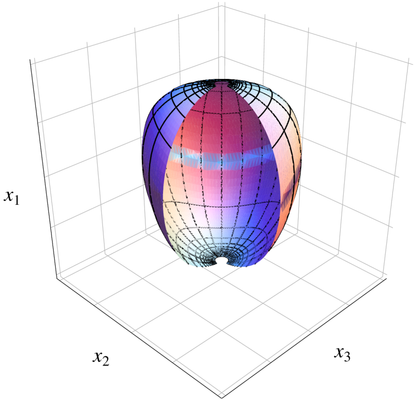
Distance:

$$d(x, y) = \inf\{\ell(\gamma(\cdot)) \, : \, \gamma(0) = x, \, \gamma(T) = y\}$$

Riemannian manifold: Heisenberg group

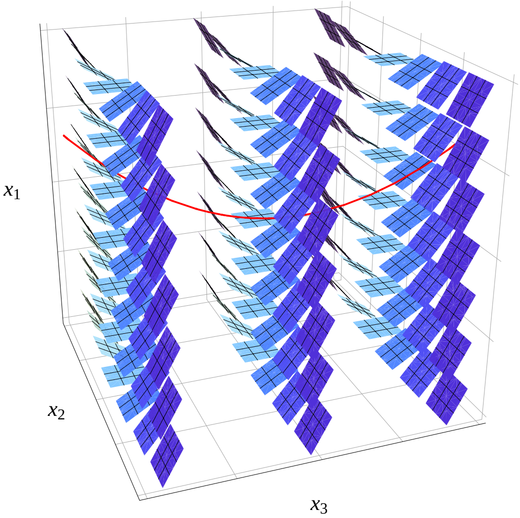


geodesics: helices and lines



$\mathcal{S}_1 = \{x \,:\, d(0,x) = 1\}$

Distribution on Heisenberg group



Distribution $\mathcal{D}$

$x \mapsto \mathcal{D}(x) \subseteq T_x\mathbb{R}^3$

(smoothly) assigns subspace to tangent space at each point

Example:
$\mathcal{D} = \mathrm{span}(\partial_{x_2}, x_2\, \partial_{x_1} + \partial_{x_3})$

Bracket generating

Sub-Riemannian manifold: Heisenberg group

Sub-Riemannian structure $(\mathcal{D}, \mathbf{g})$

- distribution $\mathcal{D}$ spanned by vector fields

$$X_1 = \partial_{x_2} \qquad \text{and} \qquad X_2 = x_2\, \partial_{x_1} + \partial_{x_3}$$

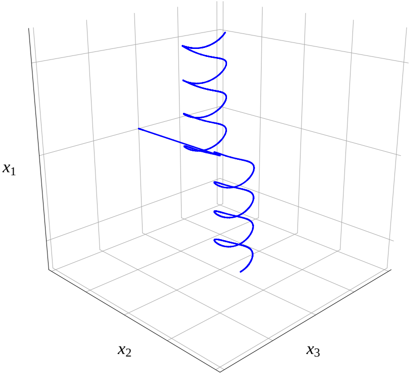
- metric $\mathbf{g} = \begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & 0 \\ -x_2 & 0 & 1 + x_2^2 \end{bmatrix}$ restricted to $\mathcal{D}$

(in fact, need only be defined on $\mathcal{D}$)

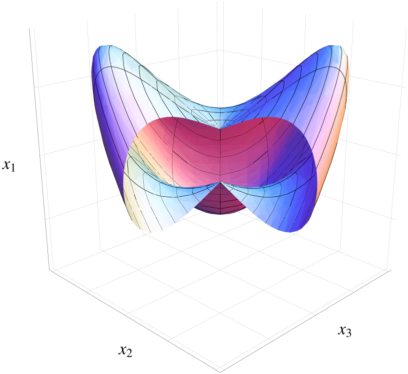
- $\mathcal{D}$-curve: a.c. curve $\gamma(\cdot)$ such that $\dot\gamma(t) \in \mathcal{D}(\gamma(t))$
- length of $\mathcal{D}$-curve: $\ell(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}(\dot\gamma(t), \dot\gamma(t))}\, dt$
- Carnot-Carathéodory distance:

$$d(x, y) = \inf\{\ell(\gamma(\cdot)) \,:\, \gamma(\cdot) \text{ is } \mathcal{D}\text{-curve connecting } x \text{ and } y\}$$

Sub-Riemannian manifold: Heisenberg group



geodesics: helices and lines



$\mathcal{S}_1 = \{x \,:\, d(0,x) = 1\}$

Invariance

Homogeneous spaces

- Group of isometries of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ acts transitively on manifold.
- For any $x, y \in \mathsf{M}$, there exists diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}$ such that $\phi_* \mathcal{D} = \mathcal{D}$, $\phi^* \mathbf{g} = \mathbf{g}$ and $\phi(x) = y$.
- On Lie groups, we naturally consider those stuctures invariant with respect to left translation.

For sub-Riemannian example we considered before

- we have transitivity by isometries:

$$\phi_a : (x_1, x_2, x_3) \mapsto (x_1 + a_1 + a_2 x_3, x_2 + a_2, x_3 + a_3)$$
$$\begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & a_2 & a_1 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}$$

Formalism

Left-invariant sub-Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

- Lie group G with Lie algebra $\mathfrak{g}$.
- Left-invariant bracket generating distribution $\mathcal{D}$
  - $\mathcal{D}(g)$ is subspace of $T_g\mathsf{G}$
  - $\mathcal{D}(g) = g\mathcal{D}(\mathbf{1})$
  - $\mathrm{Lie}(\mathcal{D}(\mathbf{1})) = \mathfrak{g}$.
- Left-invariant Riemannian metric $\mathbf{g}$ on $\mathcal{D}$
  - $\mathbf{g}_g$ is a inner product on $\mathcal{D}(g)$
  - $\mathbf{g}_g(gA, gB) = \mathbf{g}_\mathbf{1}(A, B)$ for $A, B \in \mathfrak{g}$.

Remark

Structure $(\mathcal{D}, \mathbf{g})$ on G is fully specified by

- subspace $\mathcal{D}(\mathbf{1})$ of Lie algebra $\mathfrak{g}$
- inner product $\mathbf{g}_\mathbf{1}$ on $\mathcal{D}(\mathbf{1})$.

Geodesics

The length minimization problem

$$\dot g(t) \in \mathcal{D}(g(t)), \qquad g(0) = g_0, \quad g(T) = g_1,$$
$$\int_0^T \sqrt{\mathbf{g}(\dot g(t), \dot g(t))} \to \min$$

is equivalent to the invariant optimal control problem

$$\dot g(t) = g(t) \sum_{i=1}^m u_i B_i, \qquad g(0) = g_0, \quad g(T) = g_1$$
$$\int_0^T \sum_{i=1}^m u_i(t)^2 \, dt \to \min .$$

where $\mathcal{D}(\mathbf{1}) = \operatorname{span}(B_1, \ldots, B_m)$ and $\mathbf{g_1}(B_i, B_j) = \delta_{ij}$.

Geodesics

- Via the Pontryagin Maximum Principle, lift problem to cotangent bundle $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$.
- Yields necessary conditions for optimality.

Geodesics

- Normal geodesics: projection of integral curves of Hamiltonian system on $T^*\mathsf{G}$ (endowed with canonical symplectic structure).
- Abnormal geodesics: degenerate case depending only on distribution; do not exist for Riemannian manifolds.

- Next step: minimising geodesics?

Isometries

Isometric

$(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are isometric
if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that
$\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = \phi^* \mathbf{g}'$

- ϕ establishes one-to-one relation between geodesics of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$.
- one interpretation: change of coordinates.

Isometries

Isometry group of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

$$\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\phi : \mathsf{G} \to \mathsf{G} \; : \; \phi_* \mathcal{D} = \mathcal{D}, \; \phi^* \mathbf{g} = \mathbf{g}\}$$

- represents symmetry of structure
- for invariant structures: generated by left translations on G and isotropy subgroup

$$\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \; : \; \phi(\mathbf{1}) = \mathbf{1}\}$$

Automorphisms as isometries

$\exists$ Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

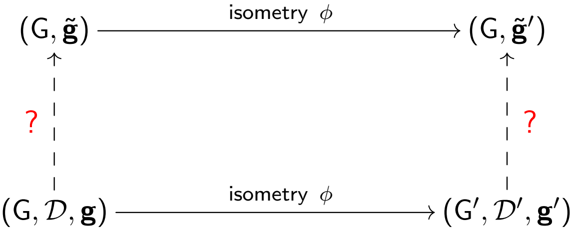
$$\phi_* \mathcal{D} = \mathcal{D}', \qquad \mathbf{g} = \phi^* \mathbf{g}'$$

if and only if

$\exists$ Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that

$$\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1}), \qquad \mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}}(\psi \cdot A, \psi \cdot B), \; A, B \in \mathfrak{g}.$$

Characteristic extensions of structures



Outlook

- Calculation of isometry groups (3D, 4D, certain classes)
- Geodesics (unified treatment)
- Riemannian case
- Affine distributions (& optimal control)