Shortest vs Straightest Curves

Sub-Riemannian Geometry and Nonholonomic Riemannian Geometry

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Outline

1. Riemannian geometry

2. Sub-Riemannian geometry

3. Nonholonomic Riemannian geometry

Introduction

Origins of Riemannian geometry

- Gauss (1777–1855) studied surfaces in $\mathbb{R}^3$
- Gauss, Bolyai (1802–1860) and Lobachevsky (1792–1856) discovered non-Euclidean geometry
- Riemann (1826–1866) made a far-reaching generalisation of these ideas, first presented in his inaugural lecture at Göttingen, 1854:

  On the Hypotheses which lie at the Foundations of Geometry

Physical interpretation

A Riemannian manifold (M, g) models a free particle moving in M (the configuration space) with kinetic energy $T(x, \dot{x}) = \frac{1}{2} g_x(\dot{x}, \dot{x})$

How to generalise Riemannian geometry?

- add constraints!
- this yields two (in general, inequivalent) geometries

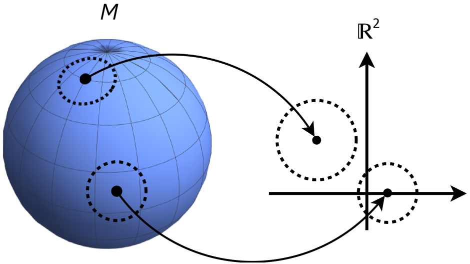
Outline

1. Riemannian geometry
2. Sub-Riemannian geometry
3. Nonholonomic Riemannian geometry

Riemannian manifold (M, g)

Smooth manifold M

- "looks like" $\mathbb{R}^n$ locally
- local coordinate system (and we can do calculus)
- tangent space $T_x M$ is the vector space of all tangent vectors at $x \in M$

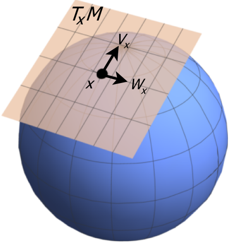


Riemannian metric g

- family of inner products:

$$g_x : T_x M \times T_x M \to \mathbb{R},$$
$$(V_x, W_x) \mapsto g_x(V_x, W_x)$$

- can define length of tangent vectors, angles between them, etc.



Examples of Riemannian geometries

Euclidean geometry

$$\mathbb{E}^n = (\mathbb{R}^n, g), \text{ where } g_x(V_x, W_x) = V_x \bullet W_x \text{ is the dot product}$$

- prototypical Riemannian manifold
- simplest case (because it's flat, i.e., the curvature $K = 0$)

Non-Euclidean geometries

- spherical geometry ($K > 0$):

$$M = \mathbb{S}^n$$

$$g = \text{dot product inherited from } \mathbb{R}^{n+1} \supset \mathbb{S}^n$$

- hyperbolic geometry ($K < 0$):

$$M = \mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$$

$$g_x(V_x, W_x) = \frac{V_x \bullet W_x}{(x_n)^2}$$

Geodesics are generalisations of straight lines to curved spaces

Motivation from $\mathbb{E}^n$: characterisations of straight lines

- shortest curve between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$:

  the line segment $\gamma : [0,1] \to \mathbb{R}^n$, $t \mapsto (1-t)x + t\,y$

- straightest curve from $x \in \mathbb{R}^n$ in the direction $V_x \in T_x M$:

  the curve $\gamma : [0,1] \to M$ with $\gamma(0) = x$, $\dot\gamma(0) = V_x$ and minimal acceleration: $\ddot\gamma = 0$

How can we define a geodesic of (M, g)?

- as shortest curves
- as straightest curves

Geodesics as shortest curves

Riemannian distance $d(\cdot, \cdot)$ on M

$$d(x, y) = \inf_{\gamma} \operatorname{length}(\gamma), \qquad x, y \in M$$

- infimum over all curves $\gamma : [0, T] \to M$ such that $\gamma(0) = x$, $\gamma(T) = y$
- $\operatorname{length}(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt$

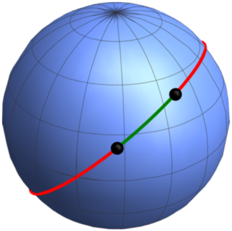
Geodesics

A curve $\gamma : [0, T] \to M$ is a

- length minimiser if

$$\operatorname{length}(\gamma) = d(\gamma(0), \gamma(T))$$

- geodesic if every sufficiently small segment is a length minimiser



Geodesics as straightest curves

Levi-Civita connection

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad (X, Y) \mapsto \nabla_X Y$$

- $\nabla_X Y$ is the "directional derivative" of Y along X
- induced by g

Acceleration

- for a curve γ, $t \mapsto \nabla_{\dot\gamma}\dot\gamma(t)$ is the acceleration of γ
- without a connection, there is no intrinsic definition of acceleration

Geodesics

A curve $\gamma : [0, T] \to M$ is a geodesic if

$$\nabla_{\dot\gamma}\dot\gamma(t) = 0, \quad \text{for every } t \in [0, T]$$

Remarks

Riemannian geodesics

- the two definitions of geodesics coincide (shortest $=$ straightest)
- geodesics are uniquely specified by initial point $+$ direction
- sufficiently close points can always be joined by geodesics

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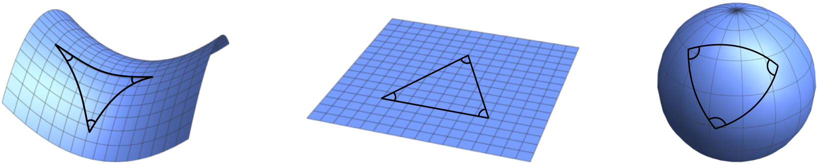
Curvature

- can use ∇ to define curvature K of (M, g)
- curvature measures how much M differs from $\mathbb{E}^n$

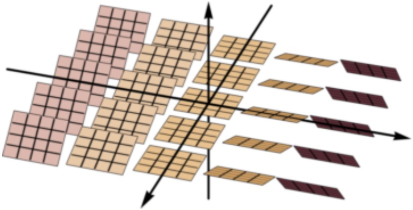


(a) $K < 0$ (b) $K = 0$ (c) $K > 0$

Generalising Riemannian geometry: constrained motion

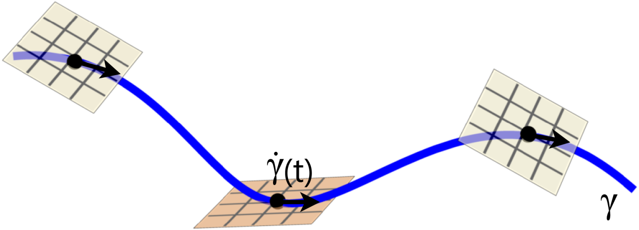
What are constraints?

- family of subspaces $\mathcal{D}_x \subset T_x M$
- $\dim \mathcal{D}_x = r$
- tangent vectors in $\mathcal{D}_x$ are "admissible velocities" from x



Admissible curves $\gamma : [0, T] \to M$

γ is admissible if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for every $t \in [0, T]$



Riemannian geometry with constraints

After introducing constraints: shortest $\neq$ straightest

We have two inequivalent geometries:

- **Sub-Riemannian geometry**
  - geodesics are shortest curves (locally length minimising)
  - fundamental object: Carnot-Carathéodory distance $d_{cc}(\cdot, \cdot)$
- **Nonholonomic Riemannian geometry**
  - geodesics are straightest curves (minimal acceleration)
  - fundamental object: nonholonomic connection $\overline{\nabla}$

Connectivity assumption

We assume (in both geometries) that any two points in M are connected by an admissible curve

Outline

Sub-Riemannian manifold $(M, \mathcal{D}, g|_{\mathcal{D}})$

Carnot-Carathéodory distance $d_{cc}(\cdot,\cdot)$ on M

$$d_{cc}(x,y) = \inf_{\gamma} \text{length}(\gamma), \qquad x, y \in M$$

- infimum over all admissible curves joining x to y
- connectivity assumption $\implies$ $d_{cc}(\cdot,\cdot) < \infty$

Sub-Riemannian geodesics

An admissible curve $\gamma : [0, T] \to M$ is a

- length minimiser if $d(\gamma(0), \gamma(T)) = \text{length}(\gamma)$
- normal sub-Riemannian geodesic if every sufficiently small segment is a length minimiser
- *abnormal* SR geodesics? — lie in the closure of $\{\text{normal geodesics}\}$

Sub-Riemannian geodesics

Remarks

- geodesics no longer specified by initial point + velocity
- uniquely specified by a covector: $\xi_x \in T_x^* M$
- sufficiently close points can always be joined by SR geodesics

Sub-Riemannian connection?

- no analog of the Levi-Civita connection in SR geometry
- as yet, no intrinsic and general notion of curvature

Some problems in SR geometry

Classification of SR structures

- particularly: left-invariant structures on Lie groups
- 3D case done; 4D case partially done

Optimal synthesis (and related problems)

- explicitly calculating SR geodesics
- given $x, y \in M$, what are the length minimisers joining x to y?
- results for structures on H_3, $SE(2)$, $SE(1, 1)$, $SO(3)$ (and some higher-dimensional Lie groups)

Outline

Nonholonomic Riemannian manifold $(M, \mathcal{D}, \mathcal{D}^\perp, g|_\mathcal{D})$

Nonholonomic connection

$$\overline{\nabla} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \quad (X, Y) \mapsto \overline{\nabla}_X Y$$

- $\mathcal{D}^\perp$ is a complement to $\mathcal{D}$ (so $TM = \mathcal{D} \oplus \mathcal{D}^\perp$)
- $\overline{\nabla}$ induced (similar to the Levi-Civita connection) by $\mathcal{D}$, $\mathcal{D}^\perp$ and $g|_\mathcal{D}$
- can define acceleration of admissible curves only

Nonholonomic geodesic $\gamma : [0, T] \to M$

An admissible curve γ is a nonholonomic geodesic if

$$\overline{\nabla}_{\dot\gamma}\dot\gamma(t) = 0, \quad \text{for every } t \in [0, T]$$

- uniquely specified by initial point + velocity
- from $x \in M$: can only reach an r-dim submanifold of points near x with NH geodesics

Curvature of nonholonomic Riemannian manifolds

Schouten and Wagner curvature

- Schouten (1928): first defined curvature of these structures
- Wagner (1935): extended Schouten's ideas; Wagner's curvature analogous to Riemannian curvature

Problems

- characterise the constant curvature spaces (in particular: flat spaces)
- holonomy groups

Some (more) problems in NH Riemannian geometry

Nonholonomic immersions and submersions

- immersions: embed one NH Riemannian structure inside another
- submersions: project one NH Riemannian structure onto another

Classification of NH Riemannian structures

- classifying left-invariant structures on Lie groups
- equivalence (but not classification) considered for 3D and some 4D structures
- 3D case: part of my PhD work

Comparing SR and NH geodesics

$(M, \mathcal{D}, \mathcal{D}^\perp, g|_\mathcal{D})$ has an associated SR manifold $(M, \mathcal{D}, g|_\mathcal{D})$

Comparison problems

Investigate the following situations:

(1) is a given NH geodesic also a SR geodesic?

(2) is a given SR geodesic also a NH geodesic?

(3) when do we have {NH geodesics} $\subset$ {SR geodesics}?

Results for (3)

- partial results, e.g., of the sort:

  Suppose $M = \mathsf{G}$ is a Lie group, and $\mathcal{D}$, $\mathcal{D}^\perp$, $g|_\mathcal{D}$ are left invariant.

  If $\mathcal{D}^\perp_\mathbf{1}$ is an ideal of the Lie algebra of G, *then* (3) *holds true.*

- research still ongoing and quite recent