Sum to Infinity - an Open-ended Investigation

Duncan Samson*

Rhodes University – Grahamstown

*With the participation of Nyasha Chiuraise, Christopher Gobane, Solomon Johnson, Phindile Mandiyasi, Mzikayise Mani, Xoliswa Mbelani, Irma Moller, May Moya, Washington Mushwana, Cheriyaparambil Raghavan, Thembile Sandi, Maud Siaw, Beauty Solani and Mzwandile Zingela.

FRF Mathematics Education Chair

Open-ended investigations are a wonderful way to access a diverse range of mathematical topics in a meaningful and engaging manner. Not only do such topics often arise unexpectedly or serendipitously, but open-ended investigations can provide an ideal context for nurturing such important dispositions as curiosity, creativity and self-confidence, along with feelings of personal relevance and a desire to engage dynamically in a process of genuine mathematical discovery.

I presented Figure 1 to a group of secondary school mathematics teachers in an enrichment programme run by the FRF Mathematics Education Chair at Rhodes University. What follows is a synthesis of some of the observations that arose from the session, along with further ideas and possible avenues for additional exploration.

![Figure 1](image)

Table 1 was completed in order to scaffold the investigation, and this was found to be an effective means of encouraging initial engagement with the task. Standard calculators generally show a maximum of only ten decimal places, but the Windows calculator set to scientific mode displays up to thirty decimal places.

<table>
<thead>
<tr>
<th>Sum showing individual terms</th>
<th>Sum as a decimal</th>
<th>Sum as a single fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{4} )</td>
<td>0.25</td>
<td>( \frac{5}{16} )</td>
</tr>
<tr>
<td>( \frac{1}{4} + \frac{1}{16} )</td>
<td>0.3125</td>
<td>( \frac{21}{64} )</td>
</tr>
<tr>
<td>( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} )</td>
<td>0.328125</td>
<td>( \frac{85}{256} )</td>
</tr>
<tr>
<td>( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} )</td>
<td>0.33203125</td>
<td>( \frac{341}{1024} )</td>
</tr>
<tr>
<td>( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} )</td>
<td>0.3330078125</td>
<td>( \frac{1365}{4096} )</td>
</tr>
<tr>
<td>( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} )</td>
<td>0.333251953125</td>
<td>( \frac{5461}{16384} )</td>
</tr>
<tr>
<td>( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} + \frac{1}{16384} )</td>
<td>0.33331298828125</td>
<td>( \frac{5461}{16384} )</td>
</tr>
</tbody>
</table>

Table 1. Sums of fractions expressed in different formats.
The teachers sat in groups comprising three or four members and were challenged to find as many patterns or interesting observations as they could in Table 1. Before reading on, take twenty minutes to try this task yourself. How many of the following did you notice?

- The sums can be expressed as \(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \ldots\).
- The sum approaches \(0.3\) as more terms are added.
- The last digit of the denominators of the individual terms alternate between 4 and 6.
- The numerator of the sum expressed as a single fraction is always 1 more than the sum of the denominators of all but the last of the individual terms. By way of example, in the fraction \(1365/4096\), \(1365 = 4 + 16 + 64 + 256 + 1024 + 1\), while in the fraction \(85/256\), \(85 = 4 + 16 + 64 + 1\).
- The sum expressed as a single fraction never simplifies, i.e. the numerator and denominator never share any common factors.
- The denominator of the sum expressed as a single fraction is always 1 more than 3 times the numerator.
- The sum of the numerator and denominator of any fraction in the third column gives the value of the numerator in the following cell. By way of example, for the fraction \(5/16\), \(5 + 16 = 21\) and 21 is the numerator of the following cell, \(21/64\). This observation leads to the following recursive formula for the sum of the first \(n\) terms:
  \[S_n = \frac{\text{Numerator of } S_{n-1} + \text{Denominator of } S_{n-1}}{4^n}\]

Expanding the last two columns of Table 1 yields a number of interesting cycling patterns. Since we have identified a number of patterns relating to the sum expressed as a single fraction, we can use these patterns to expand the final column manually and use the Windows calculator in scientific mode to express the fraction as a decimal.

**Table 2.** Sums of fractions expressed as decimals and single fractions

<table>
<thead>
<tr>
<th></th>
<th>Sum as a decimal</th>
<th>Sum as a single fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1)</td>
<td>0.25</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>(S_2)</td>
<td>0.3125</td>
<td>(\frac{5}{16})</td>
</tr>
<tr>
<td>(S_3)</td>
<td>0.328125</td>
<td>(\frac{21}{64})</td>
</tr>
<tr>
<td>(S_4)</td>
<td>0.33203125</td>
<td>(\frac{85}{256})</td>
</tr>
<tr>
<td>(S_5)</td>
<td>0.3330078125</td>
<td>(\frac{341}{1024})</td>
</tr>
<tr>
<td>(S_6)</td>
<td>0.333251953125</td>
<td>(\frac{1365}{4096})</td>
</tr>
<tr>
<td>(S_7)</td>
<td>0.33331298828125</td>
<td>(\frac{5461}{16384})</td>
</tr>
<tr>
<td>(S_8)</td>
<td>0.3333282470703125</td>
<td>(\frac{21845}{65536})</td>
</tr>
<tr>
<td>(S_9)</td>
<td>0.333332061767578125</td>
<td>(\frac{87381}{262144})</td>
</tr>
</tbody>
</table>
The sum expressed as a decimal always ends with 25.

Other than 0.25, all other decimal sums end with 125.

The digit positioned fourth from the end of the decimal sums alternates between 3 and 8, i.e. a 2-digit repeating pattern.

The digit positioned fifth from the end of the decimal sums cycles through a 4-digit repeating pattern: 2,0,7,5.

The digit positioned sixth from the end of the decimal sums cycles through an 8-digit repeating pattern: 3,2,0,9,8,7,5,4.

We could possibly conjecture that the digit positioned seventh from the end cycles through a 16-digit repeating pattern.

The last digit of the numerator of the sum expressed as a single fraction alternates between 1 and 5, a 2-digit repeating pattern.

The first digit of the numerator of the sum expressed as a single fraction cycles through a 5-digit repeating pattern: 1,5,2,8,3.

The first digit of the denominator of the sum expressed as a single fraction also cycles through a 5-digit repeating pattern: 4,1,6,2,1.

Many of these patterns are fascinating in their own right, but the mere observation of regularity is of little consequence if it cannot be developed into a potential learning experience. This is perhaps the critical challenge of any teacher wishing to use an investigative approach in the classroom. However, a simple observation often has the potential to open up a diverse range of additional topics. The challenge lies in identifying such moments and engaging with them appropriately. By way of example, take the observation made by one group of teachers that the denominator of the sum expressed as a single fraction is always 1 more than 3 times the numerator. Expressed differently, the numerator is always a third of 1 less than the denominator. Since the $n^{th}$ sum has denominator $4^n$, this means that the numerator can be expressed as $(4^n - 1)/3$. 

<table>
<thead>
<tr>
<th>$S_n$</th>
<th>$n$ths Sum</th>
<th>Denominator</th>
<th>Numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{10}$</td>
<td>0.33333301544189453125</td>
<td>$\frac{349525}{1048576}$</td>
<td></td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>0.3333332538604736328125</td>
<td>$\frac{1398101}{4194304}$</td>
<td></td>
</tr>
<tr>
<td>$S_{12}$</td>
<td>0.333333313465118408203125</td>
<td>$\frac{5592405}{16777216}$</td>
<td></td>
</tr>
<tr>
<td>$S_{13}$</td>
<td>0.33333332836627960205078125</td>
<td>$\frac{22369621}{67108864}$</td>
<td></td>
</tr>
<tr>
<td>$S_{14}$</td>
<td>0.3333333320915699005126953125</td>
<td>$\frac{89478485}{268435456}$</td>
<td></td>
</tr>
<tr>
<td>$S_{15}$</td>
<td>0.333333333022892475128173828125</td>
<td>$\frac{357913941}{1073741824}$</td>
<td></td>
</tr>
<tr>
<td>$S_{16}$</td>
<td>0.33333333325573131878204345703125</td>
<td>$\frac{1431655765}{4294967296}$</td>
<td></td>
</tr>
<tr>
<td>$S_{17}$</td>
<td>0.3333333333139307796955108642578125</td>
<td>$\frac{5726623061}{17179869184}$</td>
<td></td>
</tr>
</tbody>
</table>
Thus: \( S_n = \frac{4^n - 1}{3 \cdot 4^n} \), which can be simplified to \( S_n = \frac{4^n - 1}{3 \cdot 4^n} \), and finally to \( S_n = \frac{1}{3} - \frac{1}{3 \cdot 4^n} \).

Now, as \( n \) approaches infinity, \( \frac{1}{3 \cdot 4^n} \) approaches zero, and thus \( S_n \) approaches \( \frac{1}{3} \), which proves one of the initial observations that as more terms are added the sum approaches \( 0.\overline{3} \). What is pleasing about this outcome is that it can be arrived at from first principles, without any recourse to standard formulae for geometric series or the sum to infinity.

Having arrived at the formula \( S_n = \frac{1}{3} - \frac{1}{3 \cdot 4^n} \), it is interesting to note that it can be rewritten as \( S_n = -\frac{1}{3} \left(\frac{1}{4}\right)^n + \frac{1}{3} \), which of course is in the standard format for an exponential graph, \( y = a \cdot b^{x-p} + q \).

Thus, a graphical representation of \( S_n \) would yield the basic exponential graph \( y = \frac{1}{3} \left(\frac{1}{4}\right)^n \) reflected about the \( x \)-axis and translated vertically through \( \frac{1}{3} \) of a unit. Since this exponential graph has \( y = \frac{1}{3} \) as a horizontal asymptote, this lends visual support to the observation that as more terms are added the sum approaches \( 0.\overline{3} \). Since \( n \) can only take on discrete values (natural numbers), this also has the potential to open up a discussion on discrete versus continuous variables.

Let us now consider the observation made by another group that the numerator of the sum expressed as a single fraction is always 1 more than the sum of the denominators of all but the last of the individual terms. This can be represented symbolically as follows:

\[
S_1 = \frac{1}{4} = \frac{1}{4^1} = \frac{4^0}{4^1} \\
S_2 = \frac{5}{16} = \frac{1+4}{4^2} = \frac{4^0 + 4^1}{4^2} \\
S_3 = \frac{21}{64} = \frac{1+4+16}{4^3} = \frac{4^0 + 4^1 + 4^2}{4^3} \\
S_4 = \frac{85}{256} = \frac{1+4+16+64}{4^4} = \frac{4^0 + 4^1 + 4^2 + 4^3}{4^4} \\
\text{etc.}
\]

Using the formula for a geometric progression yields \( S_n = \frac{1(4^n-1)}{4-1} = \frac{4^n-1}{4^n} \), which is the same formula as arrived at previously.

It is perhaps also interesting to note that the series \( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots \) forms part of a class of converging series in which the first term and common ratio are not only the same, but are of the form \( \frac{1}{p} \).

For such series it can be readily shown that \( S_n = \frac{1}{p-1} \).

Although there are no doubt many more avenues to explore with this investigation, we have already touched on a diverse range of topics: rational numbers, geometric series, sigma notation, converging series and sum to infinity, conjecturing, attempting to prove conjectures, exponential functions, reflections,
translational, discrete versus continuous variables, and explicit versus recursive formulae. What is particularly meaningful is that this diverse range of topics could be accessed not only through a single investigation, but through a process of genuine mathematical discovery centred on encouraging a spirit of dynamic engagement.

Below are two further investigations centred on fractions. It is strongly recommended that before using them in a classroom setting, teachers first spend time thoroughly exploring and personally engaging with each task. This will provide critical insights into the challenges and appropriateness of each investigation. Furthermore, it will assist in identifying those magical moments which have the potential to lead to meaningful learning experiences which are the hallmark of an investigative approach.

**DENOMINATORS**

\[
\frac{1}{3} = \frac{1}{4} + \frac{1}{12}
\]

One third can be written as a sum of two unit fractions.

**FRACTIONS**

\[
\frac{2}{3} \rightarrow \frac{11}{5} \rightarrow \frac{26}{16} \rightarrow \frac{74}{42} \rightarrow \ldots
\]

In this sequence, each term is obtained from the preceding term by the rule:

\[
\frac{x}{y} \rightarrow \frac{x + 3y}{x + y}
\]

Investigate

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**TECHNO TIP**

If you don’t like having to use the mouse to insert exponents, fractions and the like in Equation Editor, here are some useful control keys that will speed up your typing.

CTRL + h produces a superscript for exponents
CTRL + l produces subscript
CTRL + 9 produces ( )
CTRL + f produces a fraction
CTRL + g allows you to type the next letter as a Greek symbol (e.g. q becomes theta, θ)
CTRL + r produces square root symbol (√)