Brombacher & Associates

Counting, manipulating numbers and solving problems

An extract from the Number Sense Workbook Series
Teacher Guide

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Developing a Sense of Number in the Early Grades

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Developing a Sense of Number

Background

As children develop their sense of number there are clearly identifiable stages/milestones: *counting all*, *counting on*, and *breaking down and building up numbers* (*decomposing, rearranging, and recomposing*).

When we observe children at work with numbers – in particular: solving problems with numbers – we can tell what stage of number development they are at.

- If we ask a child to calculate 3 + 5 and we observe that she "makes the 3" and "makes the 5" (using fingers or objects) before she combines the objects and counts all of them to determine that 3 + 5 = 8, then we say that this child is at the *counting all* stage.
- If we observe the child becoming more efficient by "making" only one of the numbers (using fingers or objects) and then counting these objects on from the other number "5: 6, 7, 8" to conclude that: 5 + 3 = 8, then we say that this child is at the *counting on* stage.
- If we observe a child, manipulating numbers to make the calculation easier, for example by saying that 8 + 7 = 8 + 2 + 5 = 10 + 5 = 15, we say that she has reached the *breaking down and building up* stage. What the child has done is to break up one of the numbers: 7 into 2 and 5 which allows her to "complete the 10" by adding the 2 to the 8 and then adding the remaining 5 to the 10 to get 15. We refer to this stage (more formally) as the *decomposing, rearranging and recomposing* stage.

It is expected that all children should reach the *breaking down and building up* stage within age appropriate number ranges.

In the early grades, we support children's development of the number concept through three distinct but interrelated activities:

- Counting,
- Manipulating numbers, and
- Solving problems.

Counting

Why is counting important?

Humans have a natural curiosity and tendency to compare:

- *Who has the most? Who has the least?*
- *Who is tallest? Who is shortest?*
- *Who is heaviest? Who is lightest?*
- *Do I have enough? Do I have too much?*
- *Which route is the longest? Which route is the shortest?*

In many situations, we can hold objects next to each other and make direct comparisons. However, when we cannot hold the things we are comparing next to each other, we need a "measuring device". Measuring devices start out informally: fingers, hands, paces and so on. With time these measuring devices become both more efficient and standardised.

As much as we compare objects, we also need to compare the quantity of a collection of objects. How many sheep? How many children? How many days? In order to answer our "how many" questions, we need to count.

Just like the shepherd in the illustration, children must *develop a number system* – they must develop the tools with which to describe quantity/'muchness'. Unlike the shepherd, however, children live in a world with an already well-defined number system. They are born into a society with an existing counting rhyme they do not have to invent their own.

The role of teachers is two-fold. On the one hand the teacher must create opportunities that reveal to children the need/importance/value of counting – counting as a way of describing the 'muchness' of quantities of objects. On the other hand the teacher must introduce children to the socially accepted counting rhyme (*social knowledge* - Piaget).

Counting is not only a tool for answering the "how many" question, it also provides the foundation for the development of number sense, arithmetic and the whole field of mathematics.

Illustration 1 – The Shepherd

Imagine a shepherd in a long forgotten time. Each morning the shepherd goes to the enclosure where the sheep are kept and he releases them for the day to go into the fields and graze. He stands at the entrance to the enclosure with his legs apart. As each sheep passes between his legs he moves a stone from the pile of stones on his left to a pile that he makes on his right.

Counting plays three important roles in the lives of children:

- Counting develops the language of number.
    - Giving meaning to the words often learnt through number songs and rhymes.
- Counting develops a sense of 'muchness' – quantity or numerosity.
    - The idea that 5 is a small number, 50 a bigger number and 500 a much bigger number, and
    - The last number 'counted' tells us the size (measure) of the group.
- Counting provides an early tool for calculating and solving problems.

Guiding the development of counting

Children come to school able to compare. The have an intuitive sense of more and less. They also have a natural interest in quantifying. There are many ways in which this interest and curiosity can be sustained through classroom activities that involve comparing and solving problems through modelling with concrete objects.

Because we want children eventually to develop more efficient (and standardised) ways of working with numbers we introduce them to the number names. We do so by teaching counting rhymes and songs. These are fun for children, especially those with actions. Number songs through their structure (pattern)

In the evening as the sheep return the shepherd reverses the process. For each sheep that passes between his legs he moves a stone from the pile he made in the morning back to the large pile. As the last sheep passes between his legs he moves the last stone back and he knows that all of his sheep are back in the enclosure. Except that there could be days when the last sheep has passed between his legs and there are still stones on the pile – on these days the shepherd knows that some of his sheep are missing – it could be that they got mixed up with his neighbour's sheep. On other days after he has moved the last stone, sheep are still coming into the enclosure – on these days, he knows that his neighbour is missing some sheep.

The shepherd is counting his sheep: for every sheep that leaves the enclosure a stone. There is a one-to-one correspondence between the sheep and the stones.

On those days that the shepherd is missing some sheep he walks over to his neighbour to see if it is possible that his missing sheep are with the neighbour. In order to show the neighbour how many sheep are missing the shepherd carries the pile of remaining stones with him.

Over time the shepherd gets tired of carrying the stones with him as he goes to visit his neighbour. To be sure he also gets tired of moving the stones every day. And so over time he invents a song – a rhyme – that he uses to "count" his sheep: one word per sheep as the sheep leave the enclosure and the same words as the sheep come back every night.

and the story they tell support the memorisation of the number names in sequence. Over time children learn to recite the number names in sequence without the songs that helped them initially. We refer to the reciting of number names in sequence as **rote counting**. Learning to rote count is a necessary early developmental stage in the development of numbers and number sense. There should, however, be no illusion in the mind of the reader, parent and teacher – rote counting (reciting the words of the number song) does not give meaning to the words of the song.

From the very first school day we begin teaching children the number names – we begin with rote counting activities.

Even while children are learning to rote count, we continue to take advantage of their natural curiosity and problem solving abilities by involving them in mathematical tasks such as comparing, sharing, breaking apart and building up. We do so by modelling situations on the mat:

- Which of these teddy bears has more counters? Which teddy bear has less?
- Who can share these counters between these teddy bears so that each teddy bear gets the same number of counters?
- Who can give this teddy bear more counters so that it has the same number of counters as the other teddy bear?
- Who can take some counters away from this teddy bear so that it has the same number of the

The shepherd is still counting his sheep: for every sheep that leaves the enclosure a word in the song/rhyme. There is a one-to-one correspondence between the sheep and the words.

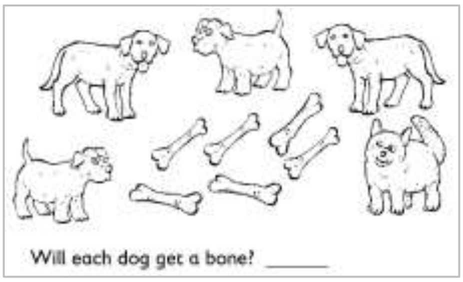
On those days that he is missing sheep he still walks over to his neighbour, but now without the stones (he has a more efficient way of knowing how many sheep he is missing). When he gets to his neighbour he recites his rhyme till he reaches the word that corresponds to the number of missing sheep and then they set about looking for the missing sheep.

More time passes and increasingly the two neighbouring shepherds start to use the same rhyme to count the sheep (standardisation) and it is no longer necessary for each shepherd to recite his rhyme to tell the other how many sheep he is missing. The shepherds can now use the unique word in the rhyme that corresponds to the number of missing sheep. The position of the word in the rhyme now suddenly represents not only the sheep that passes through the shepherd's legs at that moment, but also the collection (number) of sheep that have already passed through his legs.

The story of the shepherd illustrates how counting is a way of measuring quantities – a tool for answering the question "how many?" The story also illustrates how over time as different people work together their ways of counting become more standardised. The counting rhyme 1, 2, 3 … 9, 10, 11 … 99, 100, 101 … that we take for granted has evolved over time.

counters as the other teddy bear?

As children develop confidence with the directly modelled situations on the mat they can begin to solve similar problems present through pictures.



These problems are typically solved by one to one matching: moving counters on the mat and drawing lines on the pictures. Remember how the shepherd in the illustration “counted” by matching stones to sheep? Children are simply counting and solving problems without using numbers (yet!).

Just like the shepherd needed to develop a more efficient way of counting and solving his problem, so the child must develop more efficient ways of counting and solving problems. Children need to begin associating the words of the counting rhyme with the objects they are counting – they need to begin **rational counting**.

As children gain confidence and fluency in rote counting, we introduce the counting of objects—**rational counting**. We do not wait for children to “master” rote counting. The two types of counting build upon each other. Children will typically be able to rote count further than they can count rationally. When they are rote counting they are only concerned with the patterns of the words in the song. When they are counting rational objects they need to focus on associating the words of the song with the objects that they can count.

We expect that:

- Children can rote count in 1s up to 100 and beyond while they are still struggling to count objects up to 30,
- Children can rote count (skip count) in 10s before they can rote counts in 5s and 2s.
- Children can skip count long before they can count rational objects in groups.
- Children can rote count up to 100 and beyond and count rational objects well into the 30s before they can read and write number symbols.
- Children can more easily compare two piles in terms of which pile has more counters in it and which pile has fewer (less) counters in it long before they can compare the “size” of two numbers presented in symbols.

The important point being made is that children can do different things with numbers in different number ranges and teachers need to be sensitive to this. This is what makes the work of the teacher of mathematics in the early grades so incredibly nuanced and challenging. There are many different interrelated concepts being developed and they are developing both independently and yet in relation to each other. The teacher’s role is to know each of the children in her class and to match the range of activities to developmental states of each child.

To illustrate:

- When teachers ask children to trace line patterns they are helping children develop their writing (fine motor) skills long before their "writing has meaning". And yet, without developing their writing/fine motor skills children will never be able to write.
- When teachers ask children to trace letters and numbers they are helping children develop their skills in making these symbols long before the symbols have meaning.
- When teachers ask children to recite the number rhymes and songs (to rote count) they are helping children develop the sounds of the words long before the words have meaning.
- When teachers ask children to "count the number of objects in a collection". They rely on the child's knowledge of the number rhymes, because their counting initially associates the words of the number rhymes with objects in a one-to-one correspondence.
- When teachers ask children who can count rationally in 1s to rote count in 10s, 5s and 2s, it is because they know that the piles of counters being counted is becoming too large to count in 1s and that children will need to develop more efficient ways of counting the large pile: using groups! In order to count in groups the children will need to know (be able to recite) the skip counting songs.
- Etc.

An effective teaching and learning programme brings all of these elements together in an integrated and coordinated way.

In the next part of this chapter we describe a range of developmentally appropriate classroom activities that develop counting. In the Number Sense Workbook Series we provide written activities that support the classroom activities.

Before turning our attention to the classroom activities that support counting, there are a few general points that the teacher needs to keep in mind.

A few general comments about counting activities

- When introducing rational counting – associating the words of the number rhyme in one-to-one correspondence with the objects being counted – teachers need to:
    - Help children to work systematically.
        - When counting counters this means moving the counters as they are counted. If children do not move the counters from the pile being counted, they run the risk of counting the same counter again. This becomes all the more important as the number of counters increases.
        - When counting objects in a written context this means crossing out the objects that have already been counted.

- Choose the objects to be counted carefully. Counters should be easy to handle and should ideally all look the same. Large white beans (kidney beans) are very useful as counters. Large buttons and even bottle tops also make good counters. Building blocks are not necessarily good objects as children get distracted and want to join them together and build structures. Each teacher needs a large collection of counters in her class.

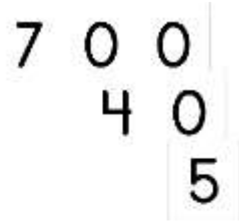
- Answering the question "How many?" represents a developmental shift for children. At first when we point to a pile of counters and ask a child to count the counters they will do so associating the words of the rhyme in one-to-one correspondence with the objects until there are no counters left (or until they start to make mistakes). If we then ask the child "So, how many counters are there in the pile?" they will often start counting the pile again: "one, two, three …". This is because they associate the question "How many?" with the process of counting. To answer "five" to the question "How many?" means that the child must recognise "five" as both the "name" of the last counter: "one, two, three, four, five" and as a description of the number of counters in the collection. The point is that teachers need to be aware of the conceptual demand that the question represents and to be patient with children as they reach of the point of seeing the double meaning of the "five" in this example. Lots and lots of counting practice will support children in making the shift.

- Writing the number symbol associated with the number of counters counted needs support. Just because children can make the symbol does not mean that they associate the symbol with the number.
  - In the lower number range (up to around 120) a number chart can be a great help. When the child has counted the objects and concluded that there are 7 counters, the teacher will hold up the number chart and ask: "Can you show me where the seven is?" If the child points to the seven, the teacher will say: "That is correct" and ask the child to write the number in her book. If the child does not know what the seven looks like, the teacher will say: "Let us look for it" she will ask the child to point to the one and then count pointing to the numbers on the number chart in sequence, counting: "one, two, three, … seven" and stop on the seven. The teacher then says: "That is what the seven looks like. Please write it down."

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140

  Not only does this help the child to learn the symbols associated with the number names, but it also provides the child with a way of 'looking for' number symbols when the child does not have the teacher to help her.

    - In the higher number ranges (from the 50s to the 1 000s) Flard cards can be both helpful in teaching children how to write numbers, while at the same time helping them to develop a better sense of how numbers break up (decompose). For example, helping children to think of 745 as 700 + 40 + 5.



    - Two cautionary comments:
        - Number charts are very helpful in developing the social knowledge of number symbols. Number charts are not helpful in helping children perform calculations. For calculations number lines are far more helpful. The problem with number charts is that while it is possible to establish that 17 + 5 = 22, the 22 is “to the left of” 17 – the relative size is not at all obvious. Teachers, however, persist in teaching children to calculate on number charts because it makes sense to them, forgetting that children do not necessarily have the same experience of the teacher.
        - As much as decomposing numbers into so-called hundreds, tens and units (as in: 745 = 7H + 4T + 5U) makes complete sense to the teacher of the early grade child, it does not make the same sense to the child. Children in the early grades learn more about the structure and meaning of number by decomposing 745 as 700 + 40 + 5 and this is supported through the effective use of Flard cards.

- Comparing numbers as in “Which number is larger 14 or 36?” is not, initially, as easy for the child as it is for the adult teacher. Comparing two piles of counters (that are obviously different in size) in terms of answering the question “Which pile has more counters?” is much easier for children. To answer the question “Which number is larger 14 or 36?” requires that a child has a sense of the ‘muchness’ of 14 and of the ‘muchness’ of 36. This ‘muchness’ is developed through rational counting activities involving estimation, counting and comparing estimates with the actual numbers. The key point is, once again, that patience is needed in helping children develop their ability to compare number symbols in terms of the size of the number. The development of this skill is supported through counting activities.

Counting activities for the classroom

Rote counting activities

Rote counting activities are appropriate for children who do not know the number names (i.e. the counting vocabulary). Typically children who start school do not know the number (counting) words or vocabulary. Before children can start counting rationally they must first learn the number (counting) words. Children learn the words best by learning the words in number 'rhymes' or 'songs'. As they learn the various number 'rhymes' and 'songs' they must become increasingly aware of the underlying patterns. In particular, they should realise that by combining the pattern 1; 2; 3; ... 9 with the pattern 10; 20; 30; … 90 they can say the number rhyme from 1 to 99.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	…						30
31	…								40

In due course when they also learn the number pattern 100; 200; 300; … 900 they can expand the rhyme to go from 1 to 999.

> Although rote counting plays an important role in developing the child's number vocabulary (i.e. the number names) children who can rote count do not necessarily associate meaning with the words.
>
> In order for the numbers to have meaning to children, they need to engage in **rational counting** activities.
>
> While it is tempting to engage the children in whole-class rote counting/chanting activities very few children benefit from this. In general, children hide in the group and the activity is nothing more than a way of keeping the class busy.

Rote counting activity 1: Counting in ones repeating after the teacher

In this activity the children repeat the number sequence after the teacher. This is necessary until the children can do so by themselves.

Activity

- The teacher says the sequence: "one, two , three, four, five"
- The children repeat the sequence after the teacher.

  - First, they do so as the small group sitting on the mat that the teachers is working with and/or as a whole class.
  - Next, they can do so as members of a small group (with or without the supervision of the teacher). All of the children in the small group repeat the sequence together.
  - Finally, they do so as individuals in a small group. As one individual recites the sequence so the others in the group listen carefully to see if they agree with the individual.
- As the children gain confidence so the teacher extends the sequence to:
  - one, two, three, … ten;
  - one, two, three, … fifteen;
  - one, two, three, … twenty; and so on.

As children gain confidence in their rote counting abilities so the teacher can expand the rote counting activities to include an increasing number range. The number “up to which children count” should increase as their confidence increases.

Rote counting activity 2: Number songs and rhymes

In addition to reciting number words after the teacher, children also benefit from learning the number words as part of a song. If the song is accompanied by actions the children become physically involved in the counting activity and this adds to their enjoyment of the activity. The purpose of the songs is to increase children’s familiarity with the number words (in sequence).

<u>Activity</u>

- Teach children to sing number songs and rhymes including the actions that accompany the words.
- Songs can be repeated frequently and added into the classroom routine. For example, some teachers teach children songs involving numbers that they sing as they enter or leave the classroom.

There are many number songs such as:

One two buckle my shoe
Three, four, knock at the door
Five, six, pick up sticks
Seven, eight, lay them straight
Nine, ten, a big fat hen
Eleven, twelve, dig and delve
Thirteen, fourteen, maids a-courting
Fifteen, sixteen, maids in the kitchen
Seventeen, eighteen, maids in waiting
Nineteen, twenty, my plate's empty

Rote counting activity 3: Counting in steps repeating after the teacher

As children gain confidence with the counting in 1s sequence, the teacher can add rhythm to the counting. This is so that children start to develop a greater sense of the many skip counting "patterns" within the number sequence. For example, when counting in 1s the teacher can draw attention to the multiples of five by having the children clap as they say each multiple: 1, 2, 3, 4, **5 (clap)**, 6, 7, 8, 9, **10 (clap)**, 11, 12, 13, 14, **15 (clap)**. The children will not recognise these numbers as multiples of five; they are simply responding to the rhythm in the number pattern and are getting ready to count in fives.

Activity

- The teacher says the sequence: "one, two, three, four, five … twenty" and she claps when she says the numbers five, ten, fifteen and twenty.
- The children repeat the sequence after the teacher clapping as the teacher did.
  - First, they do so as the small group sitting on the mat that the teacher is working with and/or as a whole class.
  - Next, they do so as members of a small group (with or without the supervision of the teacher) – with all of the children in the small group repeating the sequence and clapping together.
  - Next, they do so as individuals in a small group. As one individual recites the sequence and claps for the multiples of five so the others in the group listen carefully to see if they agree with the individual.
- As the children gain confidence the teacher repeats the activity above saying the words that are not multiples of five softer and softer until only the multiples of five can be heard.
- Children should now be able to rote count in fives
- As the children gain confidence the teacher should extend the activity to:
  - Involve increasingly larger number ranges
  - Include counting in fives, tens, twos and hundreds
  - Counting forwards and backwards first in ones then in fives, tens and twos.

Rational counting activities

As children develop their knowledge of the number names (i.e. their rote counting abilities) they become ready to start rational counting. Rational counting is the act of counting physical objects (things). Rational counting involves answering the question "How many?" For example, "How many children?", "How many cars?", "How many blocks?" etc.

Children need a lot of help as they start rational counting. They need to associate the number names of the number rhymes with the objects they are counting in a one-to-one manner. At first children will simply say the number words in sequence as they point to objects at random; they won't necessarily know when to stop counting and may even count some objects more than once.

The teacher's role is to help children organise the objects that they are counting. At first children must touch the objects as they count them. It is best if they move the objects from one place to another as they are counted.

The objects used as counters are also important. Counters should be easy to handle and all look the same. Large white beans (kidney beans) are very useful as counters. Bottle tops and large buttons also make good counters. Building blocks are not good objects because children get distracted by wanting to build structures with them. Each teacher needs a large collection of counters in her class.

As children gain confidence with rational counting, they also need to start recognising and writing number symbols. In general, children will be able to count further than they can read and write the number symbols.

Different children will be able to count objects (rational counting) correctly up to different amounts. It is best to allow each child to count as far as he/she can. Teachers should avoid deciding on a target for the whole class before the counting activities.

Rational counting activity 1: Using number cards

The activities described here are appropriate for very young children who have not yet learnt to rote count as well as for those who are starting to count rationally.

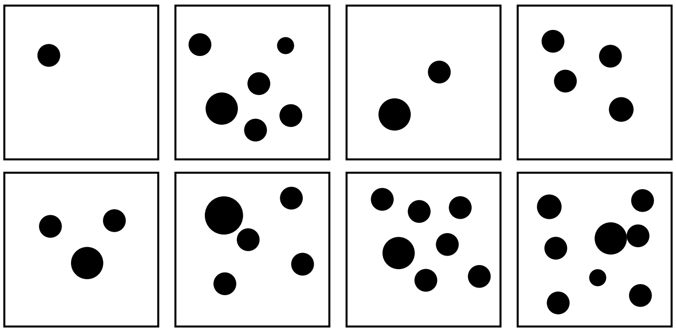
These activities also help develop visual discrimination and encourage reasoning about more, less and equal.

<u>Materials needed</u> (a print master has been supplied as an appendix to this teacher guide)

- A set of cards with dots
  - Cards should have a wide range of number of dots in different positions and of different sizes.
  - Cards with the same number of dots must have the dots in a variety of orientations.
- A set of cards with number symbols
- A set of cards with number names

Activity

The teacher should work with small groups of children at a time. The tasks described below are listed from easiest to hardest. The teacher needs to choose the task that is most appropriate to the child's developmental state.



8	5	4	2
1	7	3	6

one	seven	four	five
two	six	eight	three

- Task 1: The teacher shows the children two cards with dots on them and asks "Which card has the most dots on it?" or "Which card has the least number of dots on it?"
    - The teacher should start with two cards that have a very different number of dots on them. This is because the more similar numbers of dots on the two cards, the harder this activity is for the children. The teacher can progress to progress to cards with a more similar number of dots once they feel the child is ready.
- Task 2: The teacher gives the children a few cards and asks them to arrange the cards from the card with the least number of dots to the card with the greatest number of dots on it. Cards with the same number of dots should be placed on top of each other.
- Task 3: The teacher gives the children two sets of cards. One set of cards has dots and the other set has number symbols. The child must match the cards with dots to the corresponding cards with number symbol.
    - This task is not suitable until children know the number symbols (these are developed in the activities that follow).
    - The more cards that the children are given, the harder the task becomes.
- Task 4: The teacher gives the children three sets of cards. One set has dots, one set has number symbols and one set has number names. The children must match the cards with dots to the corresponding cards with number symbols and number names.

Rational counting activity 2: Counting small piles of counters individually

This is the first rational counting activity that children should engage in.

Materials needed

- A large set of counters. See the earlier note about the kinds of objects that make good counters.
- A large number chart with the numbers 1 to 120 or more (a print master has been supplied as an appendix to this teacher guide).
- A writing book for each child to write in.
- A pencil for each child to write with.

Activity

- The teacher arranges small groups of children (8 to at most 10) in a circle on the mat. The teacher should be a part of the circle, so that he/she can make eye contact with each child.
- The teacher places a pile of counters in front of each child. The size of the pile is determined by the teacher's knowledge of how far the children can count. Each child gets a pile of counters that is a little larger than the number to which the child can count.
- Each child in the group gets a turn to count the counters in their pile of counters. It is important that:
  - The child touches each counter as they count it. He/she should move each counter from one place to another as they count it.
  - The other children in the group listen carefully to the child who is counting and help him/her if he/she gets stuck.
  - The teacher makes a note of how far each child can count before making a mistake.
- As children gain confidence with the task the teacher can expand the activity by asking each child to point to the number that they counted to on a number chart. As a further extension the children can be asked to write the number that they counted in their writing books. These extensions are important because they contribute to children learning to read and write number symbols and names.
  - When the child has finished counting, the teacher holds the number chart up and asks the child "Can you show me where the number ** is on this chart?"
    - If the child points to the correct number the teachers points to the number and says "That is correct. That is what the number ** looks like. Everybody write down the number." All of the children write the number in their writing book.
    - If the child does not know what the number looks like, or points to the incorrect number, the teacher says "Let us look for the number." The teacher then helps the child to point to the numbers on the chart one by one while counting: "one, two, three …" and so on until the child gets to the number. At this point the teacher says "That is correct. That is what the number ** looks like. Everybody

write down the number." All of the children write the number in their writing book.

- As children gain more confidence with the task the teacher can begin to ask the question "How many counters were there in your pile?"
    - This question will not be obvious to children at first and many will start counting the pile again: "one, two, three ..." The reason for this is that children associate the counting process with the question "How many?"
    - The question also marks an important moment in the development of the child's understanding of number. The number "9" suddenly represents not only the name of the last counter counted, but also the number of counters in the pile. Teachers need to be sensitive to this developmental difficulty and must be patient with children as they work out this dual meaning of the number.

NOTE: This activity is only really suitable for children who are counting up to at most 30 counters. Once children can count rationally beyond 30 they need to move to rational counting activity 4.

Rational counting activity 3: Counting out a given number of counters

This activity is an extension of rational counting activity 2 but is listed separately because of the important difference in nature of the task.

In order for children to be able to solve problems involving numbers they need to have a sense of the meaning of numbers. Children who are at the 'counting all' stage of number development need to be able to 'make a number' in order to solve problems involving the number. Practically, this means that children need to be able to make a pile with a given number of counters in it. This is why this activity is important.

Materials needed

- A large set of counters. See earlier note about the kinds of objects that make good counters.

Activity

- The teacher arranges small groups of children (8 to at most 10) in a circle on the mat with the teacher a part of the circle, so that she can make eye contact with each child.
- The teacher places a large pile of counters in the middle of the group and one by one asks each child to make a pile with a given number of counters in it. Each child will make a pile with a different number of counters to the other children. The size of the pile that a child makes is determined by their developmental state. The teacher says to the child "Please make a pile with *11* counters in it."
    - The child must then count out the correct number of counters from the pile in the middle of the group and place these in front of them.

    - The other children must watch as the child counts out his/her pile and must offer help if they think the child has made a mistake.
- As children gain confidence in reading and writing number symbols, the teacher can change the instruction to "Please make a pile with so many counters in it" pointing to the number on her number chart and/or writing the number on a small chalkboard.

Rational counting activity 4: Estimating and counting in 1s taking turns

This activity is appropriate for children who are able to individually count objects of 30 or more. This activity also contributes to children's shift from 'counting all' to 'counting on'.

Materials needed

- A large set of counters. See earlier note about the kinds of objects that make good counters.
- A large number chart with the numbers 1 to 120 or more (a print master has been supplied as an appendix to this teacher guide).
- A writing book for each child to write in as well as a pencil for each child to write with.

Activity

- The teacher arranges small groups of children (8 to at most 10) in a circle on the mat with the teacher a part of the circle, so that she can make eye contact with each child.
- The teacher places a pile of counters in the middle of the group. She then asks the children to look at the pile of counters and to estimate how many there are. The teacher tells the children to write down their estimate in their writing books.
    - Note: the number of counters in the pile is related to the number range in which the group of children is counting. Typically the pile should be a little larger than the number up to which the children in the group can count. It is, therefore, important that the teacher knows the developmental state of each child in the group.
- After each child has written down their estimate the teacher asks each child to tell the group their estimate. After each child has told the group their estimate the teacher can ask questions such as:
    - "Whose estimate is the largest?"
    - "Whose estimate is the smallest?"
    - "Whose estimates are bigger than 20?"
    - "Whose estimates are smaller than 50?"
- The teacher now asks the first child to start counting the counters and stops the child after he/she has counted for a while (see comments below). After stopping the child the teacher asks all of the children in the group to write down in their books the number reached by the child who was counting. If the children struggle to write down the number the teacher can again use the number chart as she did in rational counting activity 2.

- After each child has written the number and the teacher has checked that they have done so correctly, the teacher asks the next child to continue counting the pile of counters. The child continues from where the previous child stopped and moves the counters that he/she counts to the pile of counted counters. The teacher asks the child stop after he/she has counted for a while (see comments below) and asks all of the children in the group to write down the number reached by the child who was counting in their books.
- The activity continues in this way until all of the counters have been counted.
- When all of the counters have been counted and the children have all written down the number of counters in the pile, the teacher can ask questions such as:
  - "Whose estimate was closest to the number of counters in the pile?"
  - "Whose estimate was furthest from the number of counters in the pile?"
  - "How far was your estimate from the actual number?"

<u>A few comments about the activity</u>:

- Each child who gets a turn to count should count at least 14 to 20 counters so that they are forced to count through at least one or two _9 to _0 transitions. The reason for this is that children typically struggle with the decade transitions but not with the numbers inside a decade once they get there. For example
  - The first child counts past 9 to 10 and beyond and past 19 to 20 and beyond before the teacher stops her at about 26,
  - The next child then counts past 29 to 30 and beyond and also past 39 to 40 and beyond before the teacher stops him at 44. And so on.
- Make sure that each child counts through at least one decade transition. For example:
  - If there are 8 children in the group and there are about 65 counters in the pile, then let each child count through two decade transitions and to let the fourth child start at the beginning again. This is better than letting each child only count a few counters but not getting an opportunity to count through the decade transitions.
- If a child makes an estimate that is far too low it is a good opportunity to address it when counting the counters. For example:
  - There are 120 counters in the pile and one of the children has estimated that there are 25 counters. The teacher, then, may stop the first child who is counting at 25 and speak to the child who estimated 25 saying "You estimated 25. Are 25 or more than 25 counters in the whole pile? Do you want to change your estimate?" This will help children to develop a stronger sense of the amounts associated with different numbers.
- It is important that the teacher plans to count in this way well beyond one hundred as soon as possible. Grade 1 children, in the second half of the year, should be able to reach this point. There are at least two reasons:

    - Firstly, it is unnatural to have an “upper limit” when counting. Counting beyond 100 assists with the development of a child’s sense of ‘muchness’ and exposes children to the pattern that underpins the structure of the counting numbers.
    - Secondly, by stopping at 100, children develop a sense that the world stops at 100 and when asked to count on from 100 say things like “98, 99, 100, 200, 300 ...” or some variation.
- As children start manipulating numbers (see the later section) teachers can enrich the counting activity by asking questions such as:
    - “How many more counters do we need to get to 60? Let’s check.”
    - “How many more counters do we need to get to 100? Let’s check.”

<u>NOTE:</u> This activity is only really suitable for children who are counting up to at most 200 counters. Once children can count rationally beyond 200 they need to move to more efficient counting strategies – namely, counting in groups (see rational counting activity 6).

Rational counting activity 5: Counting in groups – counting body parts

As children gain confidence with counting in 1s they should be encouraged to start counting in groups. The human body provides some natural groups. The body has two eyes, two ears, two hands and two feet. The body has 10 fingers, 10 toes, two groups of five fingers on each hand and two groups of five toes on each foot. The groups found on the human body are at the heart of the base ten number system. It is interesting to notice that the body does not have groups of 3 or groups of 4. Counting in groups of 3 and groups of 4 is not a natural thing to do and does not contribute to the development of children’s sense of number.

<u>Activity</u>

- After completing rational counting activity 4 for the day, the teacher asks “How many ears (or eyes, hands or feet) does Sipho have? How many ears does Mary have? How many ears does Piet have? Who can count all of the ears in the group?”
- The teacher asks a volunteer to walk around counting the ears of all the children in the group. The volunteer must stand up and walk around the group touching each of the ears that he/she counts.
    - The first volunteer will typically count “one, two, three ... sixteen”
- The teacher asks the question “Who can count the ears more quickly?” She then chooses a volunteer to count the ears of all the children in the group. The volunteer must stand up and walk around the group touching each of the ears that he/she counts.
    - The second volunteer will typically repeat what the first volunteer did, but do so more quickly.

- The teacher repeats the question: "How many ears (or eyes, hands or feet) does Sipho have? How many ears does Mary have? How many ears does Piet have? Who can use this fact to count all of the ears in the group as quickly as possible?"
  - The next volunteer will typically touch two ears at a time and count "two, four, six ... sixteen".
- At this point the teacher can ask other volunteers to repeat the counting. The process of counting in groups has been initiated. Children need a lot of practice with this activity.

A few comments about the activity:

- This rational counting activity is not an independent activity that needs a great deal of time. It is an add-on to rational counting activity 4 which should be used when children are ready to start counting more efficiently. It would probably be appropriate to start using this activity in the second half of Grade 1.
- The developmental sequence for counting in groups in this way is to first count in twos (ears, eyes, hand and feet), then in tens (fingers and toes) and only then in fives (fingers on a hand or toes on a foot).
  - When counting fingers (in tens and fives) it is important that each child in the group place their hands on the mat so that all of their fingers are visible and that the child counting the fingers touches the groups of fingers aware that they are counting a group and not single items using a "different song".

Rational counting activity 6: Counting in groups – counting large piles of counters efficiently

As children start to count with confidence into the hundreds and after they have been introduced to counting body parts in groups they are ready to start counting large piles of counters efficiently. This is an activity that will typically be used from the last few months in Grade 1 right through to the end of Grade 3 with the only difference being the size of the pile of counters being counted. This activity assumes that rational counting activities 4 and 5 are familiar to the children.

Materials needed

- A large set of counters. See earlier note about the kinds of objects that make good counters.
- Flard cards (a print master for the Flard cards has been supplied as an appendix to this teacher guide).
- A writing book for each child to write in as well as a pencil for each child to write with.
- Containers that can be used to store the counters counted on one day when the children will count further on the next day.

Activity

- The teacher arranges small groups of children (8 to at most 10) in a circle on the mat with the teacher a part of the circle, so that she can make eye contact with each child.

- The teacher places a large pile of counters in the middle of the group. She asks the children to look at the pile of counters and to estimate how many there are. The teacher tells the children to write down their estimate in their writing books.
    - Note: The number of counters in the pile is determined by the developmental state of the children in the group. For Grade 1s there might be between 70 and 130 counters. For Grade 3s there could be more than 400 counters.
- When each child has written down their estimate the teacher goes around the group and asks each child to tell the group their estimate. After each child has told the group their estimate the teacher can ask questions such as:
    - "Whose estimate is the largest?"
    - "Whose estimate is the smallest?"
- The teacher says to the children "If we counted this pile of counters in ones we would be busy all day. How could we count the pile more quickly?" The children will suggest counting the pile in twos or tens or fives (developed in rational counting activity 5). The teacher helps the children agree on the group size for counting the pile of counters.
    - Note: The teacher's response to the suggestions of the children is determined by the number of counters in the pile. For piles up to 70 or 80 counting in twos makes sense and the teacher may agree to counting in twos. For piles larger than 70 or 80, the teacher will suggest that the pile is counted in groups of 10 or 5. The teacher may also suggest that a pile is counted in groups of 10 or 5 if the she wants to practice counting in 10s or 5s.
- **If the pile is to be counted in twos** the teacher asks the children in the group to take turns counting in twos. The children should move two counters at a time, as happened when counting in 1s in rational counting activity 4. As with rational counting activity 4, the teacher stops each child after they have counted between 14 and 20 counters, the children in the group all write down the number and the next child continues until the pile has been counted.
- **If the pile is to be counted in tens or fives** the teacher asks the children in the group to take counters from the large pile and to make piles of 10 (or 5) in front of themselves. Each of the children gets involved and continues in making piles until all of the counters in the middle pile have been used up. It should be expected that there will be a few left over counters and these are left or put back in the middle of the mat.
- Once the large pile has been arranged in small piles of 10 (or five) counters the first child touches her piles counting one by one counting "ten, twenty, thirty, ... fifty" until all her piles are counted and the next child continues counting "sixty, seventy, eighty, ... one hundred and ten" until all his piles are counted. Continue in this way until all the counters have been counted.
- The children in the group use the Flard cards to make the total and write it into their writing books (see separate note on working with Flard cards).

- When all of the counters have been counted and the children have all written down the number of counters, the teacher can ask questions such as:
  - "Whose estimate was closest to the number of counters in the pile?"
  - "Whose estimate was furthest from the number of counters in the pile?"
  - "How far was your estimate from the actual number?"
- Once children have gained confidence in the activity and **if the pile was counted in tens or fives** the teacher may now encourage the children to rearrange their piles into more organised arrangements and then to count the counters again in increasingly larger groups:
  - If the pile was reorganised into groups of 10 then these groups of ten can be arranged:
    - In rows of two groups of ten so that the counters can be counted in 20s
    - In rows of five groups of ten so that the counters can be counted in 50s
    - In rows of ten groups of ten so that the counters can be counted in 100s
      - For a pile of 348 counters the teacher may help the children to see that three rows of ten piles of counters leads to 300, another four piles of counters leads to 40 and there are 8 loose counters: 348 = 300 + 40 + 8.
  - If the pile was reorganised into groups of 5 then these groups of five can be arranged:
    - In rows of two groups of five so that the counters can be counted in 10s
    - In rows of four groups of five so that the counters can be counted in 20s
    - In rows of five groups of five so that the counters can be counted in 25s
    - Etc.

<u>A few comments about the activity:</u>

- After organising the counters into rows and logical larger groups, the teacher can draw children's attention to the fact that the groups sometimes allow for counting in different ways. For example, if the counters are grouped in fives and there are there are five piles of counters in four rows, then the collection can either be counted as 25 + 25 + 25 + 25 = 100 or as 20 + 20 + 20 + 20 + 20 = 100. In other words, 4 groups of five 5s is the same as five groups of four 5s. So, 4 fives = 5 fours. By drawing attention to these relationships teachers are drawing children's attention to the commutative property of multiplication, namely that $a \times b = b \times a$.
- Counting large piles of counters in groups is important for several reasons:
  - Counting large numbers is important because it helps children to develop a sense of quantity ('muchness'). For example, unless children have worked with 748 they have no sense that 748 is a large number, that 74 is much less and that 8 is really small by comparison.
  - Counting large piles in groups makes the point that in mathematics our goal is to be efficient. 748 counters could be counted in 1s and moved them from one group to another, but that could result in errors *and* it would take a very long time. Using groups is more efficient than counting in 1s both in terms of time and in terms of accuracy.

  - Counting large piles of counters in groups that link to the base 10 number system expose children to the structure of numbers. It draws attention to the fact that 748 = 700 + 40 + 8 and that 9 748 = 9 000 + 700 + 40 + 8 etc.
- Although children can count several hundred counters very quickly in groups (as described), teachers may be concerned about starting at 1 again every day. In order to get to larger numbers more quickly, the teacher may keep the counters that were counted on one day and count on from that pile the next day. For example, if 368 counters counted on a particular day the teacher then places them in a container writing 368 on the side. The next day when the children count again the teacher can make a pile with the 368 next to another pile of counters. The children then count on from 368 to determine the number of counters in the two piles combined.

NOTE: This activity is suitable for children who can:

- Count beyond 100 in 1s with confidence, and
- Who have learnt the skip counting patterns through rote counting activities.

There is no upper limit to the number of counters that children can count in this way. By Grade 3 children should be counting several hundred counters in this way. However, it is not very important that children count counters beyond 1 000.

Manipulating numbers

Calculating fluently and efficiently

Numbers and calculations with numbers are at the heart of mathematics. Children need to develop a range of calculating strategies that enable them to calculate flexibly and fluently. Furthermore, it is important that they can perform a wide range of calculations mentally. Mental calculations are central to estimating.

It is unlikely that we would expect anybody to spend time calculating 24.382 × 0.248 using paper and pencil in a context where we have calculators. However, it is important that a person has a sense of what the expected answer is. In the case of 24.382 × 0.248 we expect a person to have a sense that 24.382 × 0.248 ≈ 12 × 0,5 = 6 so that when they use their calculator and get 6.046736 as the answer they are not surprised.

Calculating flexibly means using different calculation strategies for different situations.

Calculating fluently means confidently using a range of calculation strategies within a number range and for operations appropriate to a child's developmental state.

To illustrate flexibility consider the two calculations:

$$37 + 49 = \square \text{ and } 36 + 47 = \square$$

When performing the first calculation mentally it may make it easier to think of the calculation as:

$$37 + 49 = 37 + 50 - 1 = 87 - 1 = 86$$

This approach involves recognising that 49 is very close to 50. Adding 50 to 37 is easy, so what remains is to subtract the 1 that was added to 49 to create 50.

When performing the second calculation mentally it may make it easier to think of the calculation as:

$$36 + 47 = 36 + 4 + 43 = 40 + 43 = 83$$

This approach involves breaking up the 47 into 4 + 43. This is done because 4 is needed to "fill up the 36 to make 40," and, adding 40 to the remaining 43 is quite easy.

The illustration makes the point that the calculation strategy used to perform a calculation is chosen in terms of the numbers being calculated with. Flexibility in calculating refers to an individual's ability to *choose* an effective strategy for the calculation being performed. It goes without saying, that the calculation strategy used by a child is determined by the calculation being performed, as well as the child's developmental state, the child's confidence and their "sense of number".

Almost all calculation strategies involve breaking down, rearranging and building up numbers. We break down one or more of the numbers in a way that will make the calculation easier. We rearrange the numbers that we now have and then we build up the result. In the case of 36 + 47 = ☐:

- We broke up 47 into 43 + 4 because the 4 would help us to make 36 into 40,
- Rearranged the numbers: 36 + 4 + 43, and
- Built up the resulting 83 by first adding 36 + 4 and then adding the remaining 43.

In order for children to be able to calculate flexibly and fluently they need to develop a wide range of different number manipulation and calculating strategies. At the same time they need to have a great deal of practice in using these strategies.

We help children begin to develop the different manipulation and calculation strategies through daily mental arithmetic activities. These daily mental arithmetic activities should reveal the strategies through patterns. Daily mental arithmetic activities should be part of our daily classroom routine.

NOTE: Development of the different number manipulation and calculation strategies are the consequence of deliberately designed and carefully coordinated classroom activities. We refer to these as the <u>**number manipulation activities**</u>.

A ***mental number line*** is at the heart of mental arithmetic and calculating flexibly and fluently. A mental number line is an image in the mind of a number line that children move up and down with confidence. At first a child's number line will include only the single digit numbers. With time, however, the number line will 'grow' and children will gain confidence in moving up and down the line focusing on different parts. They will be able to 'zoom out' and see the number line stretching from 0 to 100 and they will be able to 'zoom in' to see all the fractions between 5 and 6. As children gain confidence in moving around the number line they will begin to notice that:

- In the same way that 6 is one more than 5:
  - 26 is one more than 25, and
  - 146 is one more than 145.
- In the same way that 10 is 3 more than 7:
  - 30 is 3 more than 27, and
  - 270 is 3 more than 267.
- In the same way that 4 + 5 = 9:
  - 40 + 50 = 90, and
  - 400 + 500 = 900.
- In the same way that 8 + 7 is the same as 8 + 2 + 5 = 10 + 5 = 15:
  - 68 + 7 is the same as 68 + 2 + 5 = 70 + 5 = 75, and
  - 285 + 79 is the same as 285 + 15 + 64 = 300 + 64 = 364.

We nurture the development of a child's mental number line as well as their ability to calculate fluently and flexibly through the careful use of deliberately structured activities that develop the child's confidence with:

- Single digit arithmetic,
- Arithmetic with multiples of 10, 100 and 100,
- Completing 10s, 100s and 1 000s,
- Bridging 10s, 100s and 1 000s,
- Doubling and halving, and
- A wide range of multiplication facts.

To many people, these are the so-called "basic number facts". The "number facts" that children should know in order "to do mathematics". It is true that children who do not have access to these number manipulation and calculation skills are unlikely to develop fluency and flexibility with calculating. However, it is not true that memorising the "basic number facts" means that children can apply them. Children need to know their "basic number facts" in an interrelated and integrated way. They need to "see" the patterns. Modern cognitive science talks about 'constellatory thinking'.

Recent literature on learning mathematics talks about the need for *Procedural Fluency* as one dimension of so-called *Mathematical Proficiency*. Procedural fluency is not to be confused with memorisation. Procedural fluency does, however, involve the automatisation of tasks. There comes a point in every child's mathematical life when they "know" that 5 + 3 = 8. They know it without first 'making' five and 'making' three and putting them together and counting eight. Children reach this state of 'automaticity' through frequent interaction with these mathematical relationships. For this reason, daily practice is important and the need for frequent structured manipulating number activities self-evident.

General description of manipulating numbers activities

The teacher typically works with one group of children at a time. The teacher arranges small groups of children (8 to at most 10) in a circle on the mat. The teacher should be a part of the circle, so that he/she can make eye contact with each child. As children become more confident and familiar with the activity the teacher can increase the size of the group and eventually conduct the activity with the whole class.

<u>Activity</u>

- The teacher working with the group of children:
  - Tells them to put down their pencils and to sit quietly.
  - Reminds them that they should try not to use their fingers when working out the answers to the questions,
  - Asks them not to shout out the answers to her questions and to wait to be asked to give their answer before doing so, and

- To calculate the answers to all questions even if the question is not directed at them. They may be asked for their answer if the child that the teacher asked struggled with the question or gets the answer wrong.

- The teacher then goes around the group randomly selecting children to answer her questions.
  - There are typically two ways of asking a question:
    - As a direct calculation:
      - "What is 5 plus 4?"
      - "What is half of 36?"
      - "What is double 14?"
      - "What is 167 minus 50?"
    - As an equation to be solved:
      - "What must be added to 7 to get 10?"
      - "What must taken away from 45 to be left with 38?"
      - "What must be doubled to give 72?"
- The teacher structures the questions in sets to reveal the patterns that she wants children to observe. For example:
  - When *adding and subtracting with multiples of 10, 100 and 1000*, the teacher may ask:
    - "What is 5 + 2?"
    - "And what is 500 + 200?"
    - "And what is 5000 + 2000?"
    - "And 50 + 20?"
    - "What did you notice?"
    - "What do you think 400 + 300 will be? Why do you say that?"
  - When *completing 10s and 100s*, the teacher may ask:
    - "What is 7 + 3?"
    - "And what is 27 + 3?"**
    - "And what is 67 + 3?"
    - "What must be added to 87 to get 90"
    - "And what must be added to 147 to get 150?"
    - "What did you notice?"
    - "What do you think must be added to 136 to get 200? Why do you say that?"

    ** Note how 17 + 3 is initially left out. We will come back to this when children are more confident. The reason is that "twenty-seven plus three", sounds a lot more like "seven plus three" than "seventeen plus three" does. We want children to 'see' the pattern.
- The teacher adapts the number range of the questions asked to the developmental state of the children she is busy working with.

<u>A few comments about the activity</u>:

- About the difficulty level of the questions being asked by the teacher.

- When managing the manipulating number activities teachers should bear in mind that:
    - We want children to respond quickly and confidently to the questions posed. If a child struggles then the question being asked is too difficult and the teacher needs to first try an easier version of the question.
    - Children should be discouraged from using their fingers. If the teacher notices children using their fingers then she knows that the question being asked is too difficult. If this is the case the teacher needs to try an easier version of the question.
- Throughout the activity teachers should ask children to explain how they performed a calculation; especially if a particular child answered a question with confidence or very quickly.
    - Often the teacher will notice that if a child has 'seen the pattern' and is asked to explain how they did the calculation so confidently, they will be encouraged to reflect on their thinking and to articulate it. Reflecting on and describing their thinking helps children to learn.
    - By asking a child to articulate what they did when they answered a question will not only help their thinking, but also help the other children in the group to develop their understanding. To support the other children the teacher can ask one of the children who listened to the explanation:
        - "Do you understand what your friend just said?"
            - If yes, "Can you illustrate what he did by solving this problem?" and then asking the child a question with a similar structure.
            - If no, "Shall we ask him to explain it again?"
        - "Can you use what your friend did to solve the following problem?" and then asking the child a question with a similar structure.
- Teachers often make the mistake of thinking that if they "completed 10s" today and some children saw the pattern then they know how to complete tens and don't need to revisit/practice it again. This is not the case. Children need sustained practice and therefore need to be asked the same questions again and again on a regular basis. Number manipulation needs to be a daily classroom activity and needs to last for at least 5 to 10 minutes per day.
- It is not enough that children are able to do 'single digit arithmetic', 'complete' and 'bridge 10s, 100s and 100s', etc. during the daily number manipulation slot. They also need to use it in the calculations they do and problems they solve in the rest of the mathematics lesson. The teacher has an important role to play in helping children make these links. She does so by referring to these skills whenever appropriate and by example (using these skills herself).

Specific description of the different manipulating numbers activities

Manipulating numbers activity 1: Single digit arithmetic

This activity involves the addition (and subtraction) of single digits to (and from) numbers of varying sizes. These calculations do not involve the bridging of a ten (decade).

The bridging of decades is introduced in activity 3.

There are 20 fundamental addition (and subtraction) facts that all children need to know to the point of automaticity. They are:

1 + 1 = 2			
2 + 1 = 3			
3 + 1 = 4	2 + 2 = 4		
4 + 1 = 5	3 + 2 = 5		
5 + 1 = 6	4 + 2 = 6	3 + 3 = 6	
6 + 1 = 7	5 + 2 = 7	4 + 3 = 7	
7 + 1 = 8	6 + 2 = 8	5 + 3 = 8	4 + 4 = 8
8 + 1 = 9	7 + 2 = 9	6 + 3 = 9	5 + 4 = 9

The assumptions in listing these 20 facts are that if a person knows that:

- 4 + 3 = 7 then they also know that 3 + 4 = 7. This is also known as the commutative property of addition.
- 4 + 3 = 7 then they also know that 7 – 4 = 3 and 7 – 3 = 4

The exciting thing about these facts is that they are not limited to single digit addition and subtraction with totals less than or equal to 9. As we develop the mental number lines (mentioned earlier), children develop the awareness that since 4 + 3 = 7:

- 24 + 3 and 134 + 3, and
- 40 + 30; 400 + 300 and 4 000 + 3 000

All rely on the relationship: 4 + 3 = 7.

The same can be said for subtraction since 6 + 2 = 8, it follows that:

- 8 – 6 = 2; 38 – 6 = 32 and 158 – 6 = 152, and
- 80 – 60 = 20; 800 – 600 = 200 and 8 000 – 6 000 = 2 000.

<u>Questions used by teachers to develop single-digit arithmetic</u>:

1. For young children with a primitive sense of number (late Grade R and early Grade 1):

   Start with the following questions:

   - I want you to imagine the number line. Can you see it? Now look for the number 5. Can you see it? What number comes after 5?

- I want you to imagine the number line. Can you see it? Now look for the number 4. Can you see it? What number comes before 4?

Repeat these questions with different initial numbers but always asking about the number before and after that number. Be careful only to ask questions that do not result in or bridge the 10 – that will come later.

As the children's confidence increases (after some days/weeks) change the questions to:

- I want you to imagine the number line. Can you see it? Now look for the number 7. Can you see it?
  - What number comes after 7?
  - What number is two numbers after 7?
- I want you to imagine the number line. Can you see it? Now look for the number 6. Can you see it?
  - What number comes before 6?
  - What number is two numbers before 6?

Repeat these questions with different initial numbers but always asking about the numbers one, two, three etc. before and after that number. Only increase from one before and after to two before and after etc. once children no longer need to use their fingers to 'work out' the answer. Again, be careful to only ask questions that don't result in or bridge the 10 – that will come later.

2. For young children who have gained confidence with the questions in 1 (late Grade R and early Grade 1):

   Start with the following questions:

   - I want you to imagine the number line. Can you see it? Now look for the number 7. Can you see it?
     - What number do you reach when you add one to 7?
     - What must we add to 7 to reach 8?
   - I want you to imagine the number line. Can you see it? Now look for the number 3. Can you see it?
     - What number do you reach when you subtract (take away) one from 3?
     - What must we subtract from 3 to reach 2?

   Repeat these questions with different initial numbers but always asking about the number reached by adding or subtracting one from the number. Once again, be careful to only ask questions that don't result in or bridge the 10 – that will come later.

   As the children's confidence increases (after some days/weeks) change the questions by gradually increasing, the amounts added and subtracted (from one to two, and to three and so on). Be careful to only ask questions that don't result in or bridge the 10 – that will come later.

3. For young children who have gained confidence with the questions in 1 and 2 (mid Grade 1 through Grade 3 and above):

   Ask questions in sets that reveal patterns/relationships, using the 20 number facts listed at the start of this activity. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. That means, for the sets below initially the teacher will ask the first few questions in each set and with time she will ask all of the questions in the set.

   - What is 7 + 2? And what is:
     - 27 + 2
     - 37 + 2?
     - 47 + 2?
     - 87 + 2?
     - 157 + 2?
     - 287 + 2?
     - Etc.

   - What must you add to 6 to get 8? And:
     - What must you add to 26 to get 28?
     - What must you add to 36 to get 38?
     - What must you add to 46 to get 48?
     - What must you add to 86 to get 88?
     - What must you add to 156 to get 158?
     - What must you add to 276 to get 278?
     - Etc.

   - What is 8 - 3? And what is:
     - 28 - 3?
     - 38 - 3?
     - 48 - 3?
     - 88 - 3?
     - 158 - 3?
     - 288 - 3?
     - Etc.

   - What must you subtract from to 5 to get 3? And:
     - What must you subtract from 25 to get 23?
     - What must you subtract from 35 to get 33?
     - What must you subtract from 45 to get 43?

  - What must you subtract from 85 to get 83?
  - What must you subtract from 155 to get 153?
  - What must you subtract from 275 to get 273?
  - Etc.

NOTES:

- The importance of the single digit arithmetic activities described above cannot be over emphasised! The activities that follow all rely on these single digit facts.
- Throughout these single digit arithmetic activities the teacher helps children to see the underlying pattern/structure by always starting with the basic fact and then asking the related questions. For example, even with a child whose confidence is increasing we still ask "What is 8 – 5? And, what is 68 – 5? 148 – 5? And, 798 – 5?"

Manipulating numbers activity 2: Arithmetic with multiples of 10, 100 and 1000

Arithmetic with multiples of 10, 100 and 1000 is simply an application of the single digit arithmetic developed earlier. It is remarkable how many children do not see this relationship and for them adding 100 to 200 is considered difficult because it "involves large numbers".

Ask questions in sets that reveal patterns/relationships using the 20 number facts listed at the start of the *single digit arithmetic* activity. A few examples of question sets are illustrated below.

- What is 4 + 3? And what is:
  - 400 + 300?
  - 4 000 + 3 000?
  - 40 + 30?
  - 40 000 + 30 000?
  - Etc.

Notice that the sequence of asking 400 + 300 and 4 000 + 3 000 deliberately come before 40 + 30. With our English vocabulary "four-hundred plus three-hundred" or "four-thousand plus three-thousand" are clearly related to "four plus three". "Forty plus thirty" is not as clearly related to "four plus three", because the number names don't sound as similar.

- What must you add to 4 to get 6? And:
  - What must you add to 400 to get 600?
  - What must you add to 4 000 to get 6 000?
  - What must you add to 40 to get 60?
  - Etc.

- What is 7 - 2? And what is:
  - 700 – 200?
  - 7 000 – 2 000?

  - 70 – 20?
  - Etc.

- What must you subtract from to 8 to get 5? And:
  - What must you subtract from 800 to get 500?
  - What must you subtract from 8 000 to get 5 000?
  - What must you subtract from 80 to get 50?
  - Etc.

NOTE:

- As children gain confidence with this activity it should be possible to go straight to 800 + 100 etc. without first asking 8 + 1.

Manipulating numbers activity 3: Completing 10s, 100s and 1000s …

We now introduce a further 5 fundamental addition (and subtraction) facts that all children need to know to the point of automaticity. They are:

1 + 9 = 10
2 + 8 = 10
3 + 7 = 10
4 + 6 = 10
5 + 5 = 10

The assumptions with these 5 facts is that if a person knows that:

- 3 + 7 = 10 then they also know that 7 + 3 = 10. This is known as the commutative property of addition.
- 3 + 7 = 10 then they also know that 10 – 7 = 3 and 10 – 3 = 7

Multiples of 10 (100 and 1 000) are very helpful interim targets in a calculation. This is because addition with multiples of 10, 100 and 1 000 is actually quite easy if it is seen as an application of single digit facts.

Remember the illustration at the start of this section: 36 + 47 = ☐. We added 4 to 36 so that we would get to 40 because adding 40 to 43 (what is left after taking 4 from 47) is really quite easy. We refer to the process of getting to a multiple of 10 as 'completing the 10'.

As always, we ask questions in sets that reveal patterns/relationships using the 5 new number facts listed. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. For the sets below that means that initially the teacher will just ask the first few questions in each set and with time she will ask all of the questions in the set.

- What is 6 + 4? And what is:
    - 7 + 3?
    - 3 + 7?
    - 2 + 8?
    - Etc.

- What must you add to 6 to get 10? And:
    - What must you add to 7 to get 10?
    - What must you add to 3 to get 10?
    - What must you add to 2 to get 10?
    - Etc.

- What is 8 + 2? And what is:
    - 18 + 2?
    - 28 + 2?
    - 78 + 2?
    - 238 + 2?
    - 468 = 2?
    - Etc.

- What must you add to 3 to get 10? And:
    - What must you add to 23 to get 30?
    - What must you add to 63 to get 70?
    - What must you add to 383 to get 390?
    - What must you add to 993 to get 1 000?
    - Etc.

- With older, more confident children it should be possible to ask questions such as:
    - What must you add to 3 to get 10? And:
        - What must you add to 23 to get 30?
        - What must you add to 53 to get 70?
        - What must you add to 123 to get 200?
        - What must you add to 253 to get 1 000?
        - Etc.

      In the case of the last few questions we are expecting children to answer the questions in a few steps (with time they will not even need the steps). For example
        - What must you add to 53 to get 70?
            - I must add 7 to get to 60 and then another 10 to get to 70, the answer is 17 (10 + 7).

- What must you add to 123 to get 200?
  - I must add 7 to get to 130 and then another 70 to get to 200, the answer is 77 (70 + 7).
- What must you add to 353 to get 1 000?
  - I must add 7 to get to 360 and then another 40 to get to 400, and finally another 600 to get to 1 000, the answer is 647 (600 + 40 + 7).

NOTE:

- The last few examples illustrate the importance of being able to add numbers like 600, 40 and 7 to get 647. More generally it is just as important to be able to add 34 and 40 to get 74. This is referred to as adding to and subtracting from multiples of 10. Adding to and subtracting multiples of 10 includes examples like:
  - 10 + 7 = 17, 40 + 7 = 47 and 400 + 7 = 407
  - 17 – 10 = 7, 57 – 30 = 27 and 277 – 30 = 247
  - 34 + 40 = 40 + 34 = 74 and 50 + 36 = 86
  - 240 + 34 = 274
  - 284 – 30 = 254

  In these notes we do not make a special case for this category of questions. If children know that 254 = 200 + 50 + 4, and they are confident with arithmetic using single digits and multiples of 10, 100 and 1000 then they do not experience the questions illustrated above as special.

Manipulating numbers activity 4: Bridging 10s, 100s and 1000s

When children are confident with *single digit arithmetic*, *arithmetic with multiples of 10, 100 and 1 000*, and *completing 10s, 100s and 1 000s*, they are ready for *bridging 10s, 100s and 1 000s*.

The process of bridging 10s is best illustrated by an example. Consider the addition problem 8 + 7. Children who are confident with the skills already developed will think of this problem as follows:

- What do I need to add to 8 to make 10? … 2.
- Let me break 7 into 2 + 5.
- Then 8 + 7 is the same as 8 + 2 + 5 = 10 + 5 = 15

In time, we expect children to reach a level of automaticity where they don’t have to do detailed thinking when adding/subtracting single digit numbers up to 9 + 9 = 18 and 18 – 9 = 9. However, they reach this level of automaticity only after thinking about the problem as illustrated above.

Let us illustrate this using a few more examples:

- What is 36 + 7?
  - 36 + 4 + 3 = 40 + 3 = 43

- What is 56 + 38?
    - 56 + 4 + 34 = 60 + 34 = 94
- What is 87 + 56?
    - 87 + 3 + 53 = 90 + 53 = 90 + 10 + 43 = 100 + 43 = 143
- What is 687 + 568?
    - 687 + 3 + 565 = 690 + 10 + 555 = 700 + 555 = 700 + 300 + 255 = 1 000 + 255 = 1 255
  - Or, as children gain confidence:
    - 687 + 13 + 555 = 700 + 555 = 700 + 500 + 55 = 1 200 + 55 = 1 255

The same process applies to subtraction. This is also best illustrated using a few examples:

- What is 35 – 8?
    - 35 – 5 – 3 = 30 – 3 = 27
- What is 56 – 38?
    - 56 – 6 – 32 = 50 – 32 = 50 – 30 – 2 = 20 – 2 = 18
- What is 564 – 387?
    - 564 – 4 – 383 = 560 – 60 – 323 = 500 – 300 – 23 = 200 – 20 – 3 = 180 – 3 = 177
  - Or, as children gain confidence:
    - 564 – 64 – 323 = 500 – 320 – 3 = 180 – 3 = 177

The most important observation to be made in each of illustrations above is that the thought process involves the breaking up of numbers in order to complete 10s. Breaking up of numbers follows from single digit arithmetic. This is why the activities for single digit arithmetic and completing 10s are important. They are the foundational skills of addition and subtraction involving the bridging of 10s (decades).

Ask questions in sets that reveal patterns/relationships. It is also important to ask children to explain their thinking after they respond to your questions. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. For the sets below that means that initially the teacher will just ask the first few questions in each set and with time she will ask all of the questions in the set.

- What is 6 + 4? And what is:
    - 6 + 7? Explain how you worked that out.
    - 26 + 7? Explain how you worked that out.
    - 126 + 7? Explain how you worked that out.
    - Etc.

- What must I add to 7 to get 10? And what is:
    - 7 + 5? Explain how you worked that out.
    - 37 + 5? Explain how you worked that out.

- 47 + 15? Explain how you worked that out.
- 267 + 5? Explain how you worked that out.
- 267 + 25? Explain how you worked that out.
- Etc.

- What is 10 – 2? And:
  - 15 – 7? Explain how you worked that out.
  - 35 – 7? Explain how you worked that out.
  - 35 – 17? Explain how you worked that out.
  - Etc.

NOTES:

- Children who are able to calculate in the way illustrated in this activity have reached a level of fluency and flexibility with number and operations with numbers. This makes them efficient in calculating with numbers and frees them to think about the mathematics they are doing without having to spend their energy on the manipulation of numbers.
- The level of confidence illustrated in the examples above is reached only after a long time and after deliberate and frequent practice with the activities listed above.

Manipulating numbers activity 5: Doubling and halving

Doubling and halving is at the heart of efficient multiplication and division strategies. Children who can double and halve with confidence are able to multiply and divide with understanding and efficiency.

Doubling and halving literally refers to the doubling of numbers: double 15 is 30, and halving of numbers: half of 64 is 32.

If we ask a young pre-school child to share a pile of 18 counters, equally among three dolls they will usually do something like this:

- First they will give each doll 3 counters.
- After looking at the remaining pile of counters they may realise that they do not have enough counters to give each doll another three counters and so they will give each doll two counters.
  - It is as if they have “halved” the number they gave each doll the first time around. “Halving” in this case meaning a smaller pile.
- Again, they will look at the remaining pile of counters and notice that this time they do not have enough counters to give each doll another two counters. They will then give them one each (another “halving”) and there will be no counters left.

In this illustration we notice that the child gives a large amount to each doll and then a smaller amount and another smaller amount until there is nothing left over. The child is using a process of repeated “halving”.

Consider another example: 7 845 ÷ 23 (7843 marbles shared by 23 people). One way of doing this calculation might involve the following thought process:

- Let's start by giving each of the 23 people 100 marbles each. That is 2 300 marbles.
- We are left with 7 845 – 2 300 = 5 545 marbles to share out.
- Double 2 300 is 4 600. 5 545 is more than 4 600, so let us give each of the 23 people another 200 marbles.
- We are left with 5 545 – 4 600 = 945 marbles.
- Double 46 is 92, and double 460 is 920. If we give everybody another 40 marbles, we would use 920 marbles (which is less than 945).
- We are left with 945 – 920 = 25 marbles enough to give everybody another one. And, there will be two marbles left over.
- It follows that 7 845 ÷ 23 = 100 + 200 + 40 + 1 → 341 remainder 2.

When the solution is written like this it appears very clumsy, however when it is written on paper as follows it makes a lot more sense:

7 845	
– 2 300	100
5 545	
– 4 600	200
945	
– 920	40
25	
23	+ 1
2	341

7 845 ÷ 23 = 341 remainder 2

The point of these illustrations is to make the case for the value of doubling and halving in solving problem situations involving division and multiplication.

Ask questions in sets that reveal patterns/relationships. It is also important to ask children to explain their thinking. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. For the sets below that means that initially the teacher will just ask the first few questions in each set and with time she will ask all of the questions in the set.

- What is double 4? And what is:
    - Double 30?
    - Double 34? Explain how you worked that out.
    - Etc.

- What is double 7? And what is:
    - Double 60?
    - Double 67? Explain how you worked that out.
    - Etc.
- What is half of 8? And what is:
    - Half of 20?
    - Half of 28? Explain how you worked that out.
    - Etc.
- What is half of 60? And what is:
    - Half of 10?
    - Half of 70? Explain how you worked that out.
    - Etc.

Manipulating numbers activity 6: Multiplication facts

Multiplication tables are another example of so-called “basic facts” that children must know. As with the single digit addition facts discussed earlier it is indeed important that children reach a level of automaticity with respect to multiplication. However, in the same way that the case was made earlier, it is much more important to think about how children know their multiplication facts and reach levels of automaticity than it is for them to memorise these facts.

Let us say for now that we want children to know their tables up to 12 x 12; this is (as the discussion will show) quite artificial, but let us work with this for now. If children have to remember/recite/memorise their multiplication tables then there are 144 facts to be remembered.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

If we now agree that the one-times table is trivial then there remain 131 ‘facts’.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

If we then know the commutative law of multiplication, namely that 3 x 5 = 5 x 3, then we reduce the number of facts to be remembered to 66 (already an improvement on 144):

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

a × b = b × a

Now we are ready to start "our tables". We should start with the 10 x table. Why? Because it is so easy! Furthermore, when you see the pattern you are no longer limited to 1 x 10 to 12 x 10, in fact 43 x 10 and 284 x 10 are completely accessible to a Grade 3 child (even many Grade 2 children) who 'see the pattern':

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

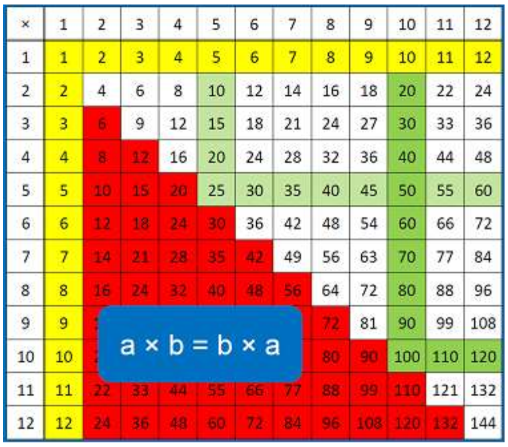
a × b = b × a

The beauty about the 10× table is that if you know the 10× table and you know how to halve then the 5× table follows. 7 × 5 is half of 7 × 10 = 70 → 35. Again, we are no longer limited to 1 × 5 to 12 × 5 because 43 × 5 and 284 × 5 are completely accessible to a Grade 3 or 4 child who 'sees the pattern':

- 43 × 5 = half of 43 × 10 = half of 430 → which is half of 400 (200) and half of 30 (15) = 215.
- 284 × 5 = half of 284 × 10 = half of 2840 → which is half of 2800 (1400) and half of 40 (20) = 1420.

And now it gets more exciting, because if you know your 10× table and the 5× table follows through halving then you also get the 15× table. For example:

- 43 × 15 = 430 (43 × 10) + 215 (43 × 5 = half 43 × 10 - 430) = 645.

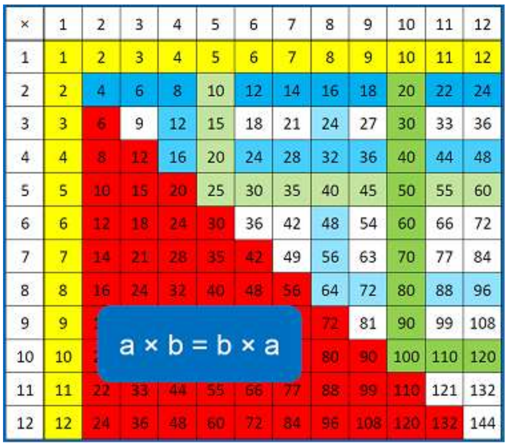


×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Next, children should know their 2× table – which is really no more than doubling. In other words doubling is the important skill (just as halving was for the 10 and 5 times table). If you can double you can multiply by 2 and by 4 (doubling again) and by 8 (doubling again). Once more if a child understands multiplying by 2 (4 and 8) as doubling they are no longer limited to 1 × 2 to 12 × 8. For example:

- 24 × 8 → 24 doubled → 48 doubled → 96 doubled again → 192.

Now notice how we are down to 21 facts still to be remembered:



×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Next children should ‘learn’:

- The 9× table: multiplying by 9 is like multiplying by 10 and taking one away
- The 11× table: multiplying by 11 is like multiplying by 10 and adding one on.

Notice we are now down to 10 multiplication facts still to be remembered:

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	a × b = b × a						72	81	90	99	108
10	10							80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

If children then learn the 3× table, the 6× table follows through doubling and they have mastered the 144 facts we started out with and more.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	a × b = b × a						72	81	90	99	108
10	10							80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Solving problems

The role of problems in learning mathematics

Mathematics is a tool for solving problems. Problems, however, also provide a way of supporting the learning of mathematics.

Children can add, subtract, multiply and divide long before they know these words. When a mother gives her children some sweets and asks them to share them equally between themselves they can do so.

Living organisms are natural problem solvers. Consider a plant growing in the ground. If the root meets a stone, it grows around the stone. When an animal senses danger, it will run away and hide or change colour or attack the perceived danger. When a young baby is hungry, he/she will cry to get attention. Children who come to school know how to solve problems – what they do not know are the labels that adults use to describe their natural responses to a problem. This is particularly true in mathematics.

When we present a young child with some toy animals and a pile of counters and ask the child to share the counters equally between the animals they will do so. Try it! You will be amazed. Young children have naturally efficient strategies for sharing the counters between the toy animals. Young children can solve this problem, and problems like it, long before they can count the counters in the pile, long before they know what it is to divide and long before they can write a number sentence to summarise the problem situation and its solution.

In a problem-driven approach to learning mathematics, we present children with problem situations that they are capable of solving and where the natural solution strategy they use is the mathematics we want them to learn in a more formal

Illustration 1: Sharing in a ratio

Working with a group of Grade 3s (8 and 9 years olds) the teacher set the scene. He said, "In today's problem there are two dogs. A small dog and a large dog. Every time that the small dog gets one biscuit the large dog gets two. If there are twelve biscuits altogether how many biscuits will each dog get?"

The group of children set to work.

Ben drew a small circle and a large circle and then drew stripes, counting as he did so: 1 (under the small circle); 2, 3 (under the large circle); 4 (under the small circle); 5, 6, (under the large circle) and so on until he ended on 11, 12 (under the large circle). Next he counted the stripes under each circle and concluded that the small dog (small circle) would get 4 biscuits and the large dog would get 8 biscuits.

Fundi drew a small circle and a large circle. Instead of drawing stripes, she wrote the numeral 1 under the small circle and the numeral 2 under the large circle. On the next line she again wrote a 1 under the small circle and a 2 under the large circle and repeated this one more time. Then she paused and counted up as follows: starting under the large circle, she said, "two, four, six" and switching to the numbers under the small circle she continued, "seven, eight, nine". Realising that she had not reached twelve yet she wrote another 1 under the small circle, and a 2 under the large circle. She then counted again and realising that she had reached 12, she added up the numbers under each circle and concluded that the small dog (small circle) would get 4 biscuits and the large way. In other words, we use a problem to provoke a natural response and that response is the mathematics we want to teach/develop. This approach is not limited to the basic operations; this approach applies throughout school mathematics.

The two illustrations make the point.

Problems in the context of a problem-driven approach to learning serve three key purposes:

1. They introduce children to the mathematics that we want them to learn.
2. They help children to develop efficient computational strategies.
3. They help children to experience mathematics as a meaningful sense-making activity.

Using problems to introduce children to the mathematics we want them to learn

Children come to school with an incredible capacity for solving problems in general. One only needs to watch children at play to realise how inventive and clever they are.

In a problem-driven mathematics classroom, we approach the teaching of mathematics as an ongoing sequence of problem solving activities. These problem activities are carefully structured to provoke anticipated/predictable responses from children. The responses of children are representations of "the mathematics" that we want children to develop.

To say all this in a different way, we have a choice when teaching mathematics – particularly so in the early grades. Either we start the lesson and say, "Today we are going to learn about ratio. Ratio is defined as …" Or, we pose a problem that provokes a reaction and we then give a name to the reaction

dog would get 8 biscuits.

The teacher observed the children working. After some time he invited Ben to show the other children how he had solved the problem. He asked Fundi to do so as well. Once they had finished explaining, the teacher asked the other children if they understood what Ben and Fundi had done.

After some discussion that included the children comparing their solutions to those of Ben and Fundi, the teacher posed a new problem. He said, "In my next problem there are two dogs. A small dog and a large dog. Every time that the small dog gets one biscuit the large dog gets three. If there are twenty biscuits altogether how many biscuits will each dog get?"

Again, the group of children set to work and the teacher observed them solving the problem. After some time the teacher asked two of the children to show their solutions to the others. He asked Sara to show her solution first. Sara had struggled on the first problem, but now, inspired by Ben's solution, she solved the problem using "Ben's method". Frank who had solved the first problem in a way that was similar to Ben's solution demonstrated his solution strategy. This time Frank used numbers instead of stripes – much like Fundi – and explained that he switched to "Fundi's method" to avoid having to draw twenty stripes. Again, the teacher asked the other children in the group if they had understood what Sara and Frank had done.

After some discussion, that included the children comparing their solutions to those of Sara and Frank, the teacher posed a new problem. He said, "In my next problem there are three dogs. A small dog, a medium dog and a large dog. Every time that the small dog gets one biscuit the medium dog gets two biscuits and the large dog gets four. If there are fifty-six biscuits altogether saying, "What we have been doing is described as sharing/dividing the biscuits in a ratio. A ratio is written as …"

The key difference in the approaches is that the problem-driven approach assumes that children have the capacity to make sense of situations. In addition, that by making sense of situations children will "do" the mathematics organically. Furthermore, by "generating" the mathematics themselves they will both experience it as more meaningful (less abstract) and with greater understanding.

The danger of this discussion is that the reader develops the impression that posing problems is enough to help children learn mathematics. This is quite incorrect! At the heart of the problem-driven approach to teaching/learning mathematics is the *deliberate* design of problems (the science of teaching). The teacher uses problems with purpose/intent.

how many biscuits will each dog get?"

The group of children set to work with different children solving the problem in different ways. Some continued to use "Ben's method" even though it was proving a little less efficient with the larger number of biscuits. Some children used "Fundi's method". Some children noticed that after giving the small dog one biscuit, the medium dog two biscuits and the large dog four biscuits they had given away seven of the fifty-six biscuits. Instead of continuing to "hand out biscuits", one, two and four at a time, they simply divided fifty-six by seven and concluded that the small dog would get eight biscuits. The medium dog would get eight times two – sixteen biscuits. The large dog would get thirty-two biscuits.

The teacher had, in effect, asked Grade 3 children to divide 56 in the ratio 1 : 2 : 4 – and they had done so with confidence.

In the early grades the teachers uses problems to introduce children to the four basic operations. The need for problems to provoke the basic operations reduces as children progress through the grades and as they are performing the basic operations with increasing confidence. The focus then shifts to problems that provoke the fraction concept. To problems that provoke calculating with fractions. To problems that provoke the development and understanding

Illustration 2: Introducing the fraction concept

Working with a group of Grade 2s (7 and 8 years olds), the teacher wanted to start introducing the children to the fraction concept. The teacher chose a well-designed problem and started, "Today's problem deals with chocolate. Do you like chocolate? Really, what is your favourite chocolate bar?" She led some discussion about chocolate bars to get the children involved in the story and to make it interesting/meaningful for the children. She also had the discussion to make sure that the idea of a chocolate bar (as a rectangular shape) was clear in the minds of the children. Next she said, "In today's problem there are two children: Yusuf and Ben. They want to share three chocolate bars equally so that there is no chocolate left over. Can you show them how they can do that?"

The group of children set to work. Because children in this class were used to working on problems, they knew what they were expected to do. They tried to make sense of the problem and, because the problem was of ratio, rate and proportion and so on.

Problem types that provoke the development of the basic operations

The challenge is to use problems purposefully. In order to do so teachers need to use well designed (purposeful) problems. When using problems to introduce the basic operations we do so by posing problems involving situations that provoke:

- Addition and subtraction like strategies through:
  - **Changing** the number of objects,
  - **Combining** two or more sets of objects, and
  - **Comparing** two or more sets of objects.
- Division-like strategies through:
  - **Sharing** objects,
  - **Grouping** objects, and
  - **Proportional sharing** of objects.
- Multiplication-like strategies through:
  - The **repeated addition** of objects/amounts, and
  - **Grid/array** like arrangements of objects.

We deliberately use these different problem types to provoke the different basic operations that we want children to learn. Each problem type can be posed in a range of different ways. The different problem types and the different ways in which they can be asked are illustrated below.

unfamiliar to them, many drew a picture of the situation.

Masixole drew two faces and three chocolate bars. Next, he drew a line from the first chocolate bar to the first face and then a line from the second bar to the second face. Then he paused for a while and thought about what to do with the remaining (third) chocolate bar. After a while he drew a line through the third bar of chocolate (as if to cross it out) and drew a small piece of chocolate next to each face.

After many of the children had grappled with the problem for a while the teacher selected three different children to show and describe their solution strategies to the others. When it was Masixole's turn, he said, pointing at his picture, "This is Yusuf and this is Ben. First I drew the three bars of chocolate and then" he said, pointing at the lines, "I gave one bar to Ben and one bar to Yusuf. Then there was one bar left over. I thought about this for a while and then decided to cut this bar into two pieces and to give Yusuf and Ben one piece each."

The teacher asked Ben, "Why did you cut the bar into two pieces?" and he responded "Because there were two children."

The teacher asked the other children in the group if they understood what Masixole and the others had done. After some discussion, that included the children comparing their solutions, she posed a two more problems:

Note: The number size in the problems that follow can be changed to suit the age and developmental state of the child.

Change problems used to provoke addition and subtraction like responses

- *Result unknown (✓ ± ✓ = ?)*
    - *Ben has 8 marbles. His father gives him 6 more marbles. How many marbles does Ben have now?*
    - *Ben has 8 apples. He eats 2 of his apples. How many apples does he have left over?*
- *Change unknown (✓ ± ? = ✓)*
    - *Ben has 8 marbles. His father gives him some more marbles. If Ben now has 12 marbles, how many marbles did his father give him?*
    - *Ben has 8 apples. He eats some of his apples. If Ben now has 5 apples left over, how many*

"Jan, Sarah and Ben want to share four bars of chocolate equally so that there is no chocolate left over. Can you show them how they can do that?" and, "Fundi, Jan, Sarah and Ben want to share five bars of chocolate equally so that there is no chocolate left over. Can you show them how they can do that?"

The children worked on the problems and the teacher observed them and asked individual children to explain to her what they were doing. Occasionally she made a suggestion to an individual child, or asked some questions that forced the child to think about what the problem was asking. The teacher noticed that Verencia drew careful representations of the problems. In particular, she noticed that Verencia's pieces from the left over bar (which she gave to the children in each problem) were getting smaller from the first problem to the last. The teacher made a note to herself that she would ask Verencia to explain to the group why she had drawn these pieces as getting smaller and smaller.

After a while the teacher once again asked, carefully selected, children to explain their solutions to the group. The teacher led a discussion and when rounding off the discussion, asked the children to summarise what they had done. The group concluded that they have given each child in the problem a bar of chocolate and if there was a left over bar they would cut it up into pieces and give each child a piece. The teacher asked, "Do you just cut the bar up into pieces?" "No" said the children, "You cut the bar into as many pieces as there are children. If there are four children you cut the bar into four pieces." The teacher asked, "So what if there are six children and one left over bar of chocolate?" "Then" said one of the children, "you cut the left over bar into six pieces."

Finally, the teachers asked Verencia, "I noticed in your pictures that the pieces were getting smaller. Why was that?" Verencia responded, "As more and more children had to share the left over bar of chocolate the pieces got smaller. The more children you share a bar of chocolate with the less each one gets."

In one problem episode the teacher had, using a well-structured series of problems, established the notion that a whole can be broken into any number of pieces/parts – fractions. The structure of the problems had also enabled Verencia to realise, in effect, that one-half is larger than one-third and that one-third is larger than one-fourth.

The teacher posed a series of problems. The children responded to the problems in a completely natural way. The response of the children was the mathematics that the teacher wanted to introduce: the fraction concept.

apples did he eat?

- *Start unknown (? ± ✓= ✓)*
    - *Ben has some marbles. His father gave him 4 more marbles. If Ben now has 12 marbles, how many marbles did he have to begin with?*
    - *Ben has some apples. He eats 2 of his apples. If Ben now has 5 apples left over, how many apples did start with?*

Combine problems used to provoke addition and subtraction like responses

- *Total unknown* (✓ + ✓ = ?)
    - There are 4 boys and 5 girls in the class. How many children are there in the class altogether?
    - Sara bakes 5 muffins. Tim bakes 8 muffins. They put the muffins together in a basket. How many muffins are there in the basket?
- *Part unknown* (✓ + ? = ✓ or ? + ✓ = ✓)
    - There are 14 children in the class. 5 are boys and the rest are girls. How many girls are there in the class?
    - Altogether Sara and Tim baked 12 muffins. If Sara baked 8 muffins, how many did Tim bake?

Compare problems used to provoke addition and subtraction like responses

- *Difference unknown* (✓ + ? = ✓ or ? + ✓ = ✓)
    - Fundi has 8 marbles. Sara has 3 marbles. How many extra marbles does Sara need to get to have the same number of marbles as Fundi?
    - Ben has 12 marbles and Frank has 7 marbles. How many marbles must Ben give away to have the same number of marbles as Frank?

Sharing problems used to provoke division-like responses

- Four friends share 12 sweets equally between them. How many sweets does each friend get?

Grouping problems used to provoke division-like responses

- A farmer has 12 apples. He puts 4 apples in a packet. How many packets can he fill?

A general note about sharing and grouping problems

The two problems used to illustrate sharing and grouping (above) both have the same mathematical structure: 12 ÷ 4 = __ . However, the way in which the problems have been stated provoke very different responses.

In the case of the **sharing** problem, the child solving the problem might think about (or draw) the 4 friends and start out by giving each friend 2 sweets – using up 8 sweets altogether. Noticing that she still has sweets left over the child might now give each friend another sweet and noticing that there are no more sweets to share among the friends she will count and establish that each friend got 3 sweets.

In the case of the **grouping** problem, the child solving the problem might think about (or draw) the 12 apples and then rearrange them in groups of four, one group at a time until there are no apples left over to put into packets.

The actions and thought processes of the child solving the sharing and the grouping problem are quite different. These differences contribute to what will one day be a richer understanding of what it means to divide. In the case of the sharing problem – the natural response is to divide by sharing out the objects a few at a time, stopping after each stage to see how many objects remain and then continuing until there are no objects left over. This thinking is what is at the heart of the so-called long division algorithm. In the case of the grouping problem – the natural response is to remove sets of objects until no objects remain and then to count the number of sets. This is to think about division as repeated subtraction (or addition). The point is that by posing problems to provoke division-like strategies, we help children develop a deeper sense of what it is to divide and, at the same time, they develop different ways of performing the calculation.

Proportional sharing problems used to provoke division-like responses

- Every time that the small dog gets one biscuit the large dog gets two biscuits. How many biscuits will each dog get if they share 12 biscuits in this way?

A note about remainders and problems that provoke division-like strategies

Some sharing and grouping situations will involve remainders. There are typically three such situations:

- Situations where the remainder does not have an impact on the answer.
    - Ben has 18 marbles. He puts 4 marbles in a bag. How many bags can he fill?

  In this case there are two remaining marbles – there is no impact on the number of bags of four that can be made. The class and teacher can decide what to do with the left over marbles, for example they may decide to "give them to the teacher".

- Situations where the remainder impacts on the answer.
    - Mr Twala can load 15 bricks on his wheelbarrow. He has to move 85 bricks to the place where he is building. How many trips must he make with his wheelbarrow?

  After filling 5 wheelbarrows (5 ×15 = 75) there are 10 remaining bricks (85 – 75 = 10) that must also be transported. So, 5 + 1 = 6 trips are needed.

- Situations where the purpose of the remainder is to provoke the development of the fraction concept.
    - Fundi and Yusuf want to share three chocolate bars equally. Show them how to do it.

  This problem was used in the illustration at the start of this section. The development of the fraction concept needs a separate discussion.

Repeated addition problems used to provoke multiplication-like responses

- Mother buys 4 bags of apples that cost R3 per bag. How much must she pay altogether?

Grid/Array like situations in problems used to provoke multiplication-like responses

- A farmer plants tomato plants. There are 4 plants in every row and 3 rows of plants. How many tomato plants are there altogether?

A note about repeated addition and grid/array like situation problems

The two problem types used to illustrate situations that provoke multiplication-like responses have the same mathematical structure: 4 × 3 = __ . However, the way in which the problems have be stated provoke very different responses.

In the case of the **repeated addition** problem, the child solving the problem might think about (or draw) the 4 bags and work out the total cost by adding R3 + R3 + R3 + R 3 = R12.

In the case of the **grid/array** like situation, the child solving the problem might think about (or draw) the 3 rows of plants with 4 plants in every row and end up with an image of a neatly organised grid/array. The grid/array will help the child to realise that she can calculate the number of plants by counting either 3 + 3 + 3 + 3 = 12 or 4 + 4 + 4 = 12. This encourages the realisation that four lots of 3 is the same as 3 lots of four. Mathematically, that 3 × 4 = 4 × 3; the so-called commutative property of multiplication.

The actions and thought processes of the child solving these problems are quite different. These different processes contribute to what one-day will be a richer understanding of what it means to multiply.

A discussion, similar to this discussion on using problems to develop the basic operations, is possible for many/most other concepts in mathematics. For now, however, we turn our attention to the second reason for using problems: helping children to develop efficient computational strategies.

Using problems to help children develop efficient computational strategies

In the same way that we use problems to provoke children to ‘do the mathematics’ that we want to teach them, so we also use problems to help children develop their age appropriate/efficient computational (calculation) strategies.

To discuss how problems help children develop efficient computational strategies let us explore the case of sharing problems. We will then show how the teacher can change the details of the problem to help children develop more efficient and age appropriate computational strategies.

Physical modelling

Even before children know number names and symbols and are able to write these, children can solve problems. They do so by means of physical modelling.

The teacher, working with a group of children, uses toys and counters to pose a problem saying “Here are three teddy bears. Who can help me by sharing these counters between the bears so that each bear gets the same number of counters?” The children will take turns to physically share the counters between the bears. The teacher will ask the other children in the group to comment on whether the child doing the sharing of the counters has done so properly.

On the one hand, physical modelling is a primitive way of solving problems – counting out counters and moving them about is not efficient, especially as the numbers involved get larger. On the other hand, it is important that children appreciate the value of physical modelling – too many children do not realise that they can draw a picture of a situation or create a model of a situation in order to help them understand the situation (the first step to solving a problem).

Physical modelling and drawings

As children develop their fine motor skills and learn to write and draw, so the teacher continues to pose problems using physical objects (teddy bears and counters). The teacher continues to ask children to solve the problem by means of modelling it physically using the objects. After the children have solved the problem by means of physical modelling the teacher will ask the children to draw a record of what they did in their books.

The shift to asking children to draw a record of the problem situation and its solution is an important first step to the child recording their thinking. At first children are likely to draw elaborate and detailed drawings. Over time and because the teacher encourages the children to work quickly, the drawing will become less elaborate and more focussed. Children will typically start to draw a stick figure instead of a detailed drawing of a person and eventually just draw a face to represent a person. Drawing pictures of the problem situation and solution, developed through physical modelling, is an important step in helping children make the shift to using drawings to represent a problem situation and then solve it.

Drawings

As children gain confidence the teacher will be able to pose a problem and ask children to "make a plan" in their books to solve the problem. Using drawings to represent and solve problems is not only a shift away from physically modelling the problem, but an important problem-solving strategy in its own right. Generally, good problem solvers will often draw a sketch of the problem situation before attempting to solve the problem. That said, it is important that the teacher does not ask children to draw a picture, but rather to "make a plan". We do not want children to get the impression that drawing pictures is what we always want.

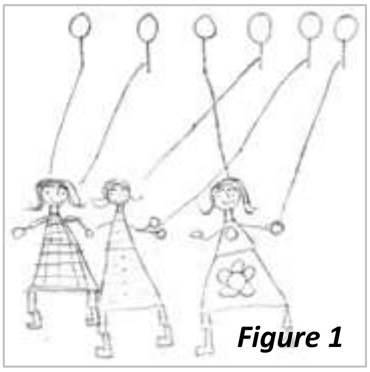


Figure 1

The ways in which children draw their solutions when solving a problem will change as the problems change. Teachers support this transition by the problems that they pose. By increasing the sizes of the numbers in the problem, the teacher provokes children to be more efficient.

Figure 1 is a drawing made by a child solving the problem "*Three friends share 6 sweets equally between themselves. How many sweets does each friend get?*" In the drawing, the child has drawn the 3 friends and the 6 sweets. The drawing shows quite clearly how the child has shared the sweets out one by one using lines to record the movement of the sweets.

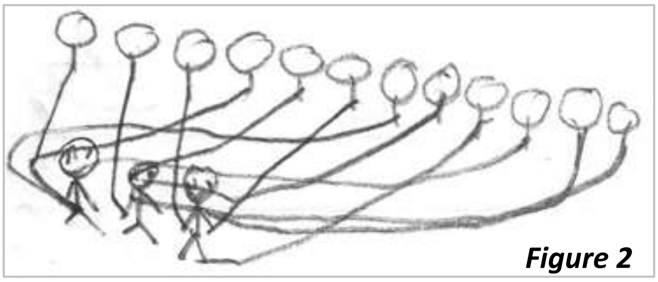


Figure 2

Figure 2 is a drawing made by a child solving the problem "*Three friends share 15 sweets equally between themselves. How many sweets does each friend get?*" This child has used a similar drawing to the one used by the child in figure 1, however because the number of sweets has increased the drawing that was appropriate for the easier problem is no longer appropriate and the child's work becomes messy and illegible. Typically, children realise that the drawing is no longer appropriate because they are starting to make errors and then they begin to use more efficient drawings.

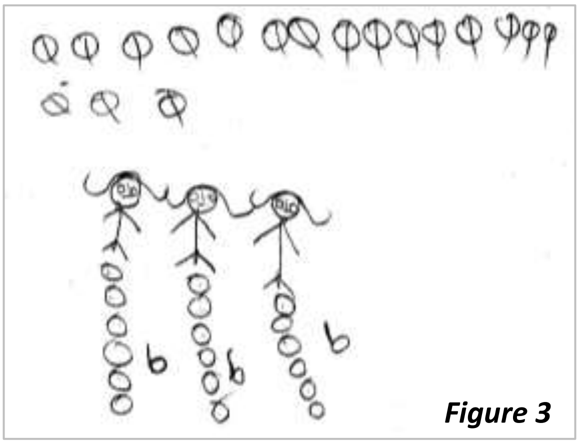


Figure 3

Figure 3 is a drawing made by a different child solving the problem "*Three friends share 18 sweets equally between themselves. How many sweets does each friend get?*" Notice how this child has drawn the friends and the collection of sweets above the friends. Instead of drawing lines to "move" the sweets as the earlier child did, this child has systematically crossed out the sweets one by one and recorded the allocation of the sweets to each friend by drawing the sweets below the friends.

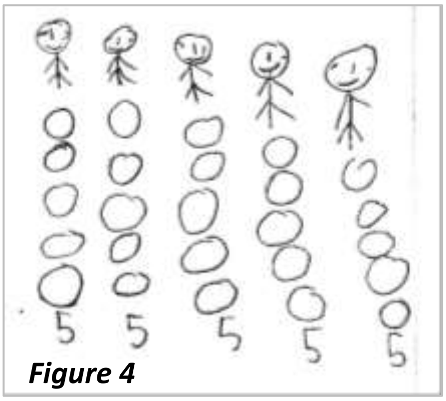


Figure 4

Figure 4 is a drawing made by another child solving the problem "*Five friends share 25 sweets equally between themselves. How many sweets does each friend get?*" Notice how this child has drawn the friends but not the collection of sweets. This child simply placed the sweets below each friend counting as she did so "one, two, three" and so on until she reached "twenty-five". This representation is more sophisticated than the previous one since the child did not first have to draw each sweet – she was able to share them out straight away.

What is evident in figures 1 to 4 is an increasing sophistication and efficiency in the drawings used by children to solve a problem. What is important to realise is that in the same way that teachers ask problems to provoke specific responses from children, so teachers also vary the sizes of the numbers in the problems to provoke children to become more efficient in their representation of the problem situation and its solution. The teacher deliberately changed the question from "3 friends and 6 sweets" to "3 friends and 12 sweets" to make the drawing used in figure 1 inefficient and in so doing to encourage children to use a more efficient drawing.

Primitive number strategies

In the same way that the type of drawing used to represent a situation and its solution became inefficient over time, so drawings also become inefficient, as the numbers in the problems get larger.

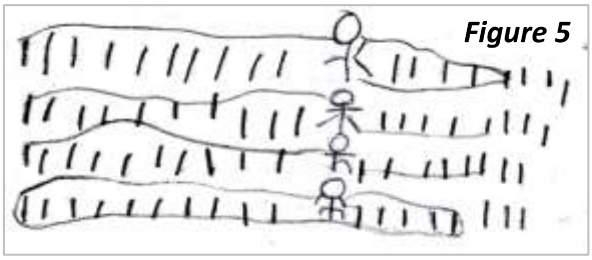


Figure 5

Figure 5 is a drawing made by a child solving the problem "*Four friends share 72 sweets equally between themselves. How many sweets does each friend get?*" The drawing does show some sophistication in that the child first drew 10 sweets for each friend (on the left) and then

5 sweets for each friend (on the right) and finally counted on from 60, allocating one sweet to each child until she reached seventy-two. However, it should be obvious that this approach is no longer appropriate for the number range of the problem.

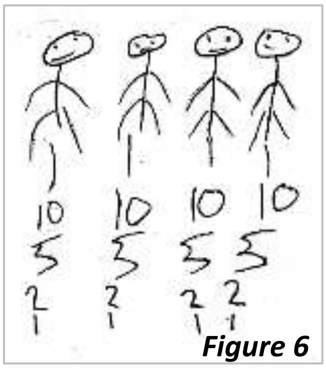


Figure 6

Figure 6 is a more appropriate response to the problem "*Four friends share 72 sweets equally between themselves. How many sweets does each friend get?*" This child has done exactly what the child in the previous problem did: he first gave each friend 10 sweets, then 5 sweets and then 2 sweets and finally one sweet each. The key difference is that this child is no longer drawing all of the sweets and is instead representing collections of them by means of numbers. The reason that the child is using numbers to represent collections of sweets is because the number of sweets in the problem is now so large that drawing individual sweets is no longer efficient or appropriate.

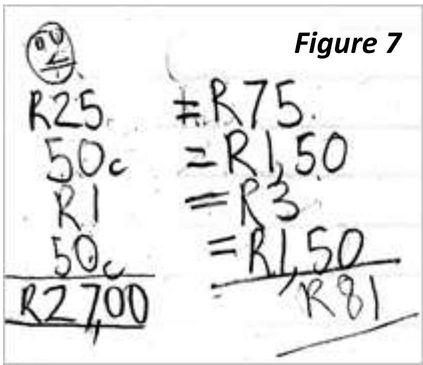


Figure 7

Efficient number strategies

As the complexity (in particular the number size) of the problems increases so children start to develop solution strategies that increasingly involve manipulating with numbers only and rely less on drawings.

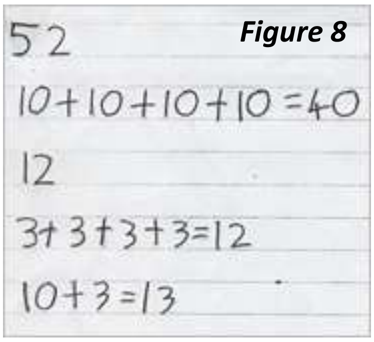


Figure 8

Figure 7 is a response by a child to the problem "*Three friends share R81 equally between themselves. How much does each friend get?*" The child does not draw each friend in the problem. Instead, he draws one friend only and gives the friend R25, he then records how giving one friend R25 amounts to giving the three friends getting R75 of the R81. He continues in this way until the R81 has been shared. In the end, he can account for the R81 that was shared out and can tell how much money each friend got. This method is largely numerical, is more efficient than using drawings nd shows an ability to manipulate with numbers.

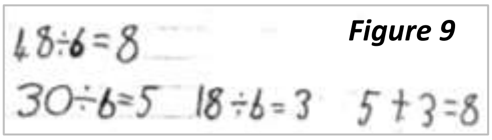


Figure 9

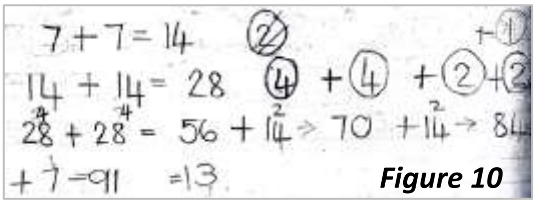


Figure 10

Figure 8 is a response by a child to a problem with the structure 52 ÷ 4. ***Figure 9*** is a response by a child to a problem with the structure 48 ÷ 6 and ***Figure 10*** is a response by a child to a problem with the structure 91 ÷ 7. These children are all solving the problem by numerical methods alone. The child solving the problem with the structure 52 ÷ 4 first adds four tens, then realises that she still needs another twelve and knowing that four threes is equal to twelve concludes that 52 ÷ 4 = 13. This child has calculated 52 ÷ 4 by building up the number. The child solving the problem 48

÷ 6 breaks up 48 into two numbers that are divisible by 6: thirty and eighteen. She divides each by 6 and concludes that 48 ÷ 6 = 8. The child solving the problem 91 ÷ 7 builds the number up through doubling to a point (56) and then building up in multiples of 7 till he gets to 91. Finally, he counts up the number of 7s used and concludes that 91 ÷ 7 = 13.

The point made in this discussion is that teachers, by asking appropriate problems, can provoke children to become increasingly efficient in their computational strategies. As the numbers used in the problems become increasingly larger and the problem increasingly complex so the early primitive strategies are no longer useful and children begin to think about more efficient solution strategies.

The role of the teacher in helping children develop more efficient calculation strategies

As has already been discussed, teachers contribute to children developing more efficient calculation strategies by managing the size of the numbers in the problem. There is, however, another very important role played by the teacher, it is: managing discussions.

When teachers set a problem for children to solve, it is important that the teacher allows enough time for children to work on the problem independently. While the children are working, the teacher observes them and, in her mind, classifies each child's approach in terms of its level of sophistication. After the children have had enough time and most of them have solved the problem the teacher then manages a discussion of the solution strategies. Typically, she will ask a child whose method is not very sophisticated (in terms of the age of the child) to explain their method so that those children who were struggling to make sense of the problem can be inspired. She will also select one or more children with more sophisticated approaches to explain their methods to the other children so that in turn those using less sophisticated methods will be inspired. The teacher may also ask a child who made a mistake to explain their thinking so that the group can discuss why that approach did not work – there is much to be gained from a discussion of mistakes.

By asking different children who used methods varying in sophistication to explain their solution strategies to the others, the teacher exposes children to increasingly sophisticated computational approaches. This exposure to more sophisticated approaches coupled with the increasing demand of the problems (resulting from the increased number ranges, etc.) encourages children to develop increasingly more efficient computational strategies.

Using problems to help children experience mathematics as a meaningful sense-making activity

Problems give purpose to the mathematics that children learn. Children who learn mathematics through solving problems experience mathematics as a purposeful, meaningful activity. They can see the value in the mathematics that they learn. They experience mathematics as a tool – a tool for solving problems.

By contrast, children who experience mathematics only as the memorisation of facts, rules and procedures used to determine the answers to questions that make no sense, do not see the purpose in what they are doing. They then experience mathematics as confusing, frustrating and mysterious.

Managing problem solving activities in class

Solving a problem, or series of problems, is part of the everyday routine of the early grade mathematics lesson. For that matter, problem solving should be at the heart of mathematics lessons in all grades.

The problem solving activity typically involves the following stages:
- The teacher poses the problem,
- Children work on the problem while the teacher monitors their progress, and
- The teacher manages a discussion of the solutions (including mistakes) made by the children.

The teacher poses the problem

- The teacher chooses a problem mindful of the mathematics that she wants the problem to provoke (see problem types above).
- The teacher makes sure that the number range of the problem is appropriate for the developmental state of the children. With younger children, she will use smaller numbers and with children whose confidence is greater she will use larger numbers. The teacher will also adjust the number range of the problem to be in line with the computational strategies that the children are using.
- When posing the problem to the children she will say "*Today I have another problem that I want you to solve. I want you to solve it in a way that makes sense to you and I want you to be ready to explain what you did as you attempted to solve the problem. Are you ready?*" Next, she sets the scene and explains the problem, making sure that each of the children in the group understands the question being asked. Once she has finished posing the problem she encourages the children to work on the problem.

Children work on the problem while the teacher monitors their progress

- In a classroom with a healthy problem solving culture children know that they must:
    1. Think about/understand the problem,
    2. Make a plan to solve the problem,
    3. Solve the problem, and finally
    4. Think about whether their solution makes sense in the context of the problem.
- As children work on the problem the teacher will monitor what they are doing. She will:
    - Encourage those who are stuck by asking them questions such as:
        - "Tell me what the problem is in your own words."
        - "Have you solved a problem like this before? What did you do then?"

        - "What do you know? What do you want to know? How can what you know help you to solve the problem?"
    - Ask children, who are working, questions such as:
        - "Why did you do this?"
        - "How do you know that you can do what you have done?"
        - "Does your answer make sense? Can you check your answer?"
    - Make sure that the children have access to the resources they may need for the problem solving strategy they are likely to use. For example, counters for those using counters, etc.
- Throughout the problem solving time the teacher observes each child as they are working and she classifies each child's approach in terms of:
    - Whether it is likely to work in determining a solution or not.
    - How sophisticated it is in terms of the age and developmental stage of the child.
- As the children are working, the teacher thinks about who she will ask to explain their solution method to the other children in the group. Typically she will choose one or more children whose method are not very sophisticated (in terms of the age of the child) to explain their method so that those children who were struggling to make sense of the problem can be inspired. She will also select one, or more, children with more sophisticated approaches to explain their methods to the other children so that they will inspire those using less sophisticated methods. The teacher will also ask children who made mistakes to explain their thinking so that the group can discuss why that approach did not work – there is much to be gained from a discussion of mistakes.

Children discuss their solutions (including mistakes)

- After the children have had enough time to work on the problem, the teacher will ask them all to stop working and to get ready to listen to each other's solutions.
- The classroom culture needs to be such that the children know to listen carefully to each other making an effort to understand each other's explanation.
- The teacher then asks the children, she has identified, to explain their methods to the other children. As the children explain what they did she encourages the engagement of the other children by asking questions such as:
    - "So do you understand what he did?
        - If yes: "Please explain what he said in your own words?"
        - If no: "Listen carefully, we will ask him to explain again."
    - "How is what he did similar to and different from what you did?"
- Depending on the time available the teacher will pose an additional problem – the additional problem will often be of a similar structure so that children can consolidate their understanding by doing a similar/related problem.