Facilitating conceptual engagement with fractions through suspending the use of mathematical terminology

2 authors:

Pamela Vale
Rhodes University

Mellony Graven
Rhodes University

PME 42

July 3-8, 2018 --- Umeå, Sweden

Proceedings

Of the 42nd Conference of the International Group
for the Psychology of Mathematics Education

Editors: Ewa Bergqvist, Magnus Österholm,
Carina Granberg, and Lovisa Sumpter

Volume 4

Research Reports Pr – Z

FACILITATING CONCEPTUAL ENGAGEMENT WITH FRACTIONS THROUGH SUSPENDING THE USE OF MATHEMATICAL TERMINOLOGY

Pamela Vale and Mellony Graven

Rhodes University

In this study we sought to establish whether an instructional sequence focused on fractions as measures was effective in supporting a group of South African Grade 3 students' understanding of fractions. The sequence is centred on a story that utilises 'nonsense' words to describe fractions. The students in this study had already been introduced to fraction terminology and symbols, but struggled in the initial lessons of this sequence and in the pre-test to use these with understanding. This paper focuses on how this sequence's suspension of the use of the mathematical terminology in favour of these 'nonsense' words helped to facilitate students' deep engagement with the concept of fractions during these lessons.

In this paper we describe the results of a study investigating the effectiveness of an instructional sequence with the aim of facilitating students' understanding of the inverse order relation of unit fractions. This sequence of four lessons, designed by Cortina, Višňovská and Zúñiga (2012), proposes an alternative starting point to teaching fractions: using fractions as measures rather than equipartition as the context in which the concept of fractions is introduced.

The lesson sequence is centred on a story about the origins of standardised measurement. A particularly notable feature of the lesson sequence is the use of 'nonsense' words, words with no established meaning, to describe fractions rather than the accurate mathematical terminology. For students who have not previously been exposed to the mathematical names for fractions, this sequence delays the use of the mathematical vocabulary. For the students in this study, it represented a suspension of the use of the terminology they had already encountered in the vocabulary and symbols of fractions at school. The focus of this paper is on the influence that this choice to suspend the use of accepted mathematical terminology had on the engagement of the students in the lesson activities and we propose that this played an important role in the progress students demonstrated in reasoning about the relative sizes of fractions.

In order to show this, we reflect on moments of interaction with the students as they worked through the activities in the lesson sequence. We focus in particular on several moments in which students grappled with the use of the mathematical terminology related to fractions (e.g. 'half' and 'quarter') and argue that in these moments the effort to encourage accurate use of the words detracted from the desired focus of the task at hand. We contrast this with the work of the students after the 'nonsense' terminology

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 331-338). Umeå, Sweden: PME.

was introduced through the story. In addition, we report on the results of the pre- and post-tests to show how students progressed in understanding of the relative sizes of unit fractions. Significantly, this improvement was based on an assessment that utilised the mathematical vocabulary and symbols and not the 'nonsense' words and created symbols used in the story and the activities.

USING MEASUREMENT AS A CONTEXT FOR TEACHING FRACTIONS

Cortina, Višňovská and Zúñiga (2014) argue that equipartition can be a didactical obstacle to teaching fractions. They explain that equipartition has been incorrectly considered by many to be "either the only or the most advantageous way to introduce students to the topic" (Cortina, et al., 2014). Specifically, they identify three fixed images of fractions that an equipartition approach develops: "fraction as a result of acting on an object (fraction as fracture); fractions as 'so many out of so many'; [and] fraction included in a whole (pp. 4-5). This approach is limited when students need to "find meaning in uses of fractions that are inconsistent with these images" (p. 7).

In order to support students in reasoning about the relative size of fractions, Cortina and Višňovská and Zúñiga (2012) have proposed an alternative starting point to teaching fractions: using 'comparing' instead of 'fracturing'. Their resulting instructional design, which uses length measurement activities as the vehicle for fraction learning, was the focus of this research. There is ongoing research into the effectiveness of this design that points to the value of taking such an approach (see Cortina & Visnovska, 2016).

As Lamon (2012) explains, when students start working with natural numbers, measurement takes its simplest form in the counting of separable objects. When they begin to encounter fractions, the measurement of continuous quantities becomes possible (Lamon, 2012). This is done by segmenting the quantity to form whole units, then subdividing the whole units and iterating the resulting part units in order to measure. In subdividing the unit into fractional pieces, the degree of precision of the resulting measurement is increased (Lamon, 2012). Measurement contexts can thus provide particularly fertile ground for developing the concept of fractions. It is on this activity of subdivision of whole units into fractional part units that Cortina et al.'s (2012) instructional design rests.

THE ROLE OF WORDS IN CONCEPT FORMATION

Vygotsky (1987) writes "direct instruction in concepts is impossible [and leads to] mindless learning of words" (p. 170). Concept formation, he explains, involves all basic intellectual functions and is impossible without the use of words as signs or "functional tools" (Vygotsky, 1986, p. 107) that drive the formation of concepts. Development of the physiologically based intellectual processes, e.g. memory or perception, does not lead to higher forms of intellectual ability. It is verbal thinking that is necessary for the qualitatively "radical change" (p. 109) that makes thinking in concepts, such as fractions, possible.

He distinguishes between phases that lead to thinking in real concepts and argues that transition from one stage to the next is reliant on a child's verbal interaction with adults (Vygotsky, 1986). In the first phase, syncretic heap, a child groups objects randomly and words do not hold stable meanings (Vygotsky, 1986). This can be seen in the students' seemingly random use of the words 'half' and 'quarter' to describe any part of a whole during the first lesson. As Berger (2006) explains, children use words that they initially do not fully comprehend, but as they use it in communicating with adults, the meaning of the word and its associated concept evolves. In other words, the concept "undergoes substantial development for the child as [they] use the word or sign in communication with more socialised others" (Berger, 2005, p. 155). In mathematics, Berger (2005) argues, the individual is required to construct the concept such that its meaning agrees with how it is used in the mathematics community.

Berger (2006) advocates for activities that allow for idiosyncratic uses of mathematical words and symbols in the early stages of concept formation. She explains (p. 17):

> …[it] is not *how* (emphasis in original) the student uses the signs but rather *that* (emphasis in original) [they] use the signs. Through this use, the student gains access to the 'new' mathematical object and is able to communicate (to better or worse effect) about it. And…it is this communication with more knowledgeable others which enables the development of a personally meaningful concept whose use is congruent with its use by the wider mathematical community

Berger therefore seemingly argues that it is necessary that the mathematically accepted terminology must be used to allow students to come to a whole understanding of the concept of fractions. Vygotsky's notion of 'signs', however, can be understood to be broader than one specific word per concept. He writes that signs can be understood as an "auxiliary means of solving a given psychological problem" (1978, p. 52). For example, a word can be used to aid someone in remembering something and in this way the "sign acts as an instrument of psychological activity" (p. 52). It should hold therefore that if the mathematical vocabulary becomes a stumbling block to students' conceptual engagement in a task, the introduction of a new 'sign' such as, in the case of our study, a nonsense word that can serve the same psychological purpose as the original word could allow this to be overcome. Once the concept itself is therefore better formed, the original word can be substituted back, but with more conceptual clarity. We propose that this could facilitate conceptual development supporting more accurate use of the accepted terminology.

METHODOLOGY

The broader study, of which this paper represents a part, took a design research approach, as developed by Gravemeijer and Cobb (2013). Our goal was to explore the "innovative learning ecology" (p. 75) proposed by Cortina et al. (2014) in their instructional sequence. Accordingly, our retrospective analysis presented here, focuses of the use of vocabulary in the classroom in order to offer a proposal as to how the sequence works to support students' learning (Gravemeijer & Cobb, 2013).

Three South African Grade 3 classes, of 36 students each, participated in the instructional sequence, which was facilitated by the first author, with the second author in attendance for two of the lessons. The four lessons were run during the normal school day, one on each of four consecutive days. The lessons were video recorded for later analysis and the first author maintained a journal of field notes to record additional observations.

Critical incident analysis (Flanagan, 1954; Butterfield, Borgen, Amundsen & Maglio, 2005) was used to identify excerpts of the video recordings that were relevant to the research focus. These were transcribed in rich detail for further analysis. In this study we selected the moments in which fraction terminology and nonsense terminology were used by the students. In addition, students completed a pre-test and post-test assessing their understanding of the inverse size order relation of fractions. Their responses were summarised and analysed for recurring patterns in the type of errors made. Each item was analysed for patterns in responses, and each student's work was analysed for shifts in performance from pre-test to post-test. There were 83 students who completed both the pre-test and the post-test.

We recognise that in having chosen to position the first author as facilitator of the sequence, the credibility of the findings could be questioned. To offset this risk, a rich audit trail is available for scrutiny that includes the lesson videos as well as the students' test scripts. The design research approach itself also enhances the credibility in that it necessitates a strictly scripted lesson delivery. This removes much of the subjectivity in decision-making within the lesson. Furthermore, the students' test responses were analysed in addition to the lesson transcripts as a form of triangulation.

THE LESSON SEQUENCE

The overarching goal of the lesson sequence is that students come to make sense of the inverse order relation of unit fractions. Each lesson in the sequence works towards achieving this. In the first two lessons, students explore measuring length using their bodies and are prompted to think about how this differs when using small or large units. By the end of the first lesson, they should be aware of the challenge in communicating measurements when using body parts of different sizes to carry out the measurements. This leads to the second lesson, in which students come to recognise that it is more suitable to measure with a standardised unit. Students are provided with sticks of identical lengths with which to measure. Through the activities, students usually become aware that there is a remaining space not covered by a whole unit, and experience the challenge of finding a way to accurately communicate the length of this remainder.

During these two lessons, students attempted to use fraction names to describe the remainders. This was expected as the students were familiar with fraction names, but their use of these, and their assumption as to the accuracy of their descriptions, limited the realisation of the lesson aims. As an example, when asked to measure the length of their (identical) desks with the stick, the students needed to make the observation that the number of whole units measured was the same for all groups, but that the stick did not allow for an accurate answer as to how long the remainder was. What transpired was that students were convinced that they were communicating the length with precision by using a fraction name to label the remainder part (many said half, when a quarter was the closest unit fraction describing the remainder as a fraction of the stick). The lesson momentarily took a turn towards assisting the students in naming the remainder appropriately in terms of the fraction name.

As an example, one group of students entered into the following exchange with the second author:

MG:	And, how long is your desk?
Student 1:	Four and a half.
MG:	Is it a full half?
Student 1:	Yes.
MG:	How much is a half? [holds stick out to student and student touches the stick at approximately half of its length] Yes! Was it that much?
Together:	[as they re-measure the desk together] One…two…three…four…
MG:	[pointing to the remaining length of the desk] So, is it a full half?
Student 1:	No.

This exchange was similarly repeated with other students and groups of students. This was not, however the aim of the sequence, nor a part of its design, so the focus was quickly turned to simply acknowledging that there was much disagreement and no accurate way of finding and describing the remainder. In this way the desired consensus was reached that a better system than a single stick was needed

It is in the second lesson that students are told a traditional story in which an ancient potter experiences difficulty in measuring to make her pots accurately and visualises using a standard stick to measure rather than body parts. In the third lesson, the story continues with the character finding a solution to the problem of measuring the remainder by carefully constructing subunits of the stick to be used to measure the remaining lengths. In the story, these subunits are given special names – what we refer to as 'nonsense' words. The sub-units are called 'obele', or 'smalls', each with special characteristics. An 'otibele', translated as 'a small of two', is a length that fits exactly twice into the length of the stick. This is a 'half' but it is never referred to as such. A list of nonsense words with the translations was given to each student (from a small of two to a small of ten representing all unit fractions from $^1/_2$ to $^1/_{10}$).

The students adopted this terminology with delight and excitement as indicated by their repeatedly saying the words aloud and smiling as they read them. Their confused use of the fraction names entirely disappeared for the remaining lessons. It is at this point that the students became engaged in particularly rich conceptual work with fractions. They constructed units that fit exactly *x* times into the unit. For example, they constructed a single unit that would fit exactly twice into the length of the stick. Students were given straws so they could easily through trial and error cut them to the needed length. The straws were shorter than the length of the stick unit, so that students could not simply fold them in half to judge the length of a small of two. They repeated this process until they had a set of 9 subunits of decreasing size. The students were continually prompted to realise that the more times a unit fits into the whole, the smaller it is.

In the fourth lesson, students used their unit sticks and this set of smaller subunits to measure objects and realised that this solved the problem of measuring remainder lengths and communicating the lengths. Until this point, the word ‘fraction’ was not mentioned. Students took joy in naming the lengths using the nonsense words and did so in a conceptually accurate manner. They indicated understanding that a ‘small of ten’ was smaller than a ‘small of nine’.

At the conclusion of the lesson sequence students’ attention was drawn to the fact that they had been working with fractions, and that these subunits could also be named using the accepted mathematical fraction names. This was merely mentioned, and was not explored further. However, as is shown in the following section, the students made great gains in their standard test performance despite all conceptual work having been done using nonsense terminology.

ASSESSMENT RESULTS

Students showed impressive gains in their understanding of the inverse order relation of unit fractions when their pre- and post-test responses were compared.

One item asked the following: “Thembi gets half ($^1/_2$) a candy bar and Angi gets one-fifth ($^1/_5$) of a candy bar. Colour in how much candy they each get.” This item was accompanied by two identical rectangles representing these candy bars for the students to colour in. A response was considered correct if the portion coloured in was within one-tenth of the accurate amount. In the pre-test, 32 students drew the half and fifth correctly.

There were 17 students who coloured in a fraction that was more than one-tenth too large or too small, although these students correctly indicated that the half was larger than the fifth. Twenty students reversed the size order relation, indicating that a fifth was larger than a half. In the post-test, of the 17 students who were inaccurate in their drawings, only 4 students over- or underestimated by more than one-tenth. Most notable were the twelve students who reversed the size order relation in the pre-test, but who drew the fractions correctly in the post-test.

This item was followed by a question asking which child received more, Thembi or Angi. There were many students who did not answer this question (only 27 answered this in the pre-test, and 59 in the post-test), however of those who answered, an interesting observation was made that pointed to some of the confusion the students had regarding the inverse order relation. Half of the students in the pre-test provided answers that were incongruent with their drawings. That is, if their drawing showed that Angi received more, they indicated on this question that Thembi received more. This was the case for 4 students. Interestingly, eleven of the students who drew the fractions correctly named Angi as the child receiving more. In the post-test, only 3 of these students persisted in this error.

The final item on the test asked students to circle the fraction which was larger of a set of four pairs of fractions: $^1/_2$ or $^1/_4$; $^1/_5$ or $^1/_3$; $^1/_4$ or $^1/_8$; and $^3/_4$ or $^3/_3$. The figure below shows the number of students, in the pre-test and the post-test, answering correctly. A dramatic increase is evident, and there were only 15 of the 83 students who persisted in making the same errors in both tests.

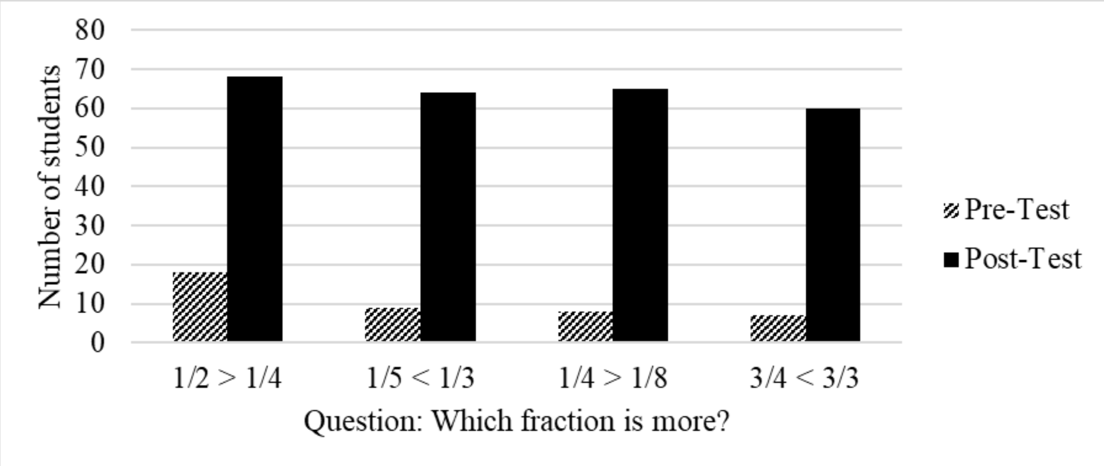

Figure 1: Number of students answering correctly.

This increase provides clear evidence of the students having improved in their understanding of the inverse order relation of unit fractions.

CONCLUSION

During the first two lessons, prior to the introduction of the nonsense words, it was clear from the students' use of the mathematically accepted fraction names, that their understanding of the concepts underlying the words was still developing. Their use of the words revealed their syncretic heap thinking (Vygotsky, 1986) and while this is a part of the development of the concept, it detracted from the work of this specific lesson sequence.

The design called for the students to suspend their use of the mathematically accepted terminology for the duration of the conceptually-driven activities of the third and fourth lesson. It was significant to observe, therefore, that the students improved, not only in their understanding of the inverse order relation of unit fractions, as was the goal, but that they were also able to demonstrate this understanding on an assessment that used the standard terminology and symbols.

This suggests that there was value in suspending the use of standard terminology, and temporarily replacing it with words that were not linked to any emerging conceptual knowledge, to allow students to engage in deep conceptual work independent of their grappling with the definitions of the technical terminology. The fraction concept itself became better formed for the students such that the original words could be substituted back and, as the students' assessment responses indicated, with increased conceptual clarity.

The use of this nonsense terminology can therefore be understood to be a feature of this lesson sequence that contributed to facilitating conceptual development that supported more accurate use of the accepted terminology.

References

Berger, M. (2005). Vygotsky's theory of concept formation and mathematics education. In H.L. Chick & J.L. Vincent (Eds.), *Proc. 29th Conf. of the Int. Group for the Psychology of Mathematics Education* (pp. 153-160). Melbourne, Australia: PME.

Berger, M. (2006). Making mathematical meaning: From preconcepts to pseudoconcepts to concepts. *Pythagoras, 63*, 14-21.

Butterfield, L.D., Borgen, W.D., Amundsen, N.E. & Maglio, A.T. (2005). Fifty years of the critical incident technique: 1954-2004 and beyond. *Qualitative Research, 5*(4), 475-497.

Cortina, J.L., Višňovská, J. & Zúñiga, C. (2012). Alternative starting point for teaching fractions. In J. Dindyal, L.P. Cheng & S.F. Ng (Eds.), *Proc. 35th Conference of the Mathematics Education Research Group of Australasia* (pp. 210-217). Singapore: MERGA.

Cortina, J.L., Višňovská, J. & Zúñiga, C. (2014). Equipartition as a didactical obstacle in fraction instruction. *Acta Didactica Universitatis Comenianae Mathematics, 14*, 1-18.

Cortina, J.L. & Višňovská, J. (2016). Reciprocal relations of relative size in the instructional context of fractions as measures. In C. Csikos, A. Rausch & J. Szitanyi (Eds.), *Proc. 40th Conf. of the Int. Group for the Psychology of Mathematics Education* (pp. 179-186). Szeged, Hungary: PME.

Flanagan, J.C. (1954). The Critical Incident Technique. *Psychological Bulletin, 51*(4), 327-358.

Gravemeijer, K. & Cobb, P. (2013). Design research from the learning design perspective. In Plomp, T. & Nieveen, N. (Eds.), *Educational Design Research* (pp. 72-113). Enschede: Netherlands Institute for Curriculum Development.

Lamon, S.J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd ed.). London: Lawrence Erlbaum Associates.

Vygotsky, L. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, MA. Harvard University Press.

Vygotsky, L. (1926/1986). *Thought and Language*. Cambridge, MA: The MIT Press.

Vygotsky, L. (1987). The development of scientific concepts in childhood. In R.W. Rieber (Ed.), *The collected works of L.S. Vygotsky: Volume 1, Problems of General Psychology* (pp. 167-242). New York: Plenum Press.