Lectures 2 & 3 - Population ecology mathematics refresher

To ease the move into developing population models, the following mathematics cribsheet is supplied. If in doubt – read a mathematics textbook!

1. Exponents and logarithms

$$a^0 = 1$$
$$a^1 = a$$
$$a^n = a.a.a.a.\text{(n times)}$$
$$a^{-n} = \frac{1}{n}$$
$$a^{p/n} = \sqrt[n]{ap} = \left(\sqrt[n]{a}\right)^p$$
$$a^p.a^q = a^{p+q}$$
$$a^{pq} = \left(a^p\right)^q$$
$$\frac{a^p}{a^q} = a^{p-q}$$
$$(ab)^x = a^x b^x$$
$$\prod_n e^{x_i} = e^{\sum_n x_i}$$

$$\ln 1 = 0$$
$$\ln e = 1$$
$$\ln b + \ln c = \ln(bc)$$
$$\ln b - \ln c = \ln\left(\frac{b}{c}\right)$$
$$\ln b^n = n \ln b$$
$$e^{\ln b} = b$$
$$a^b = e^{b \ln a}$$

2. Calculus

Differentiation

To calculate the rate-of-change of a function with respect to variable we use differentiation. This amounts to calculating the slope of the tangent to the function. Alternatively we can estimate the maximum value of a function with respect to a variable of interest.

In its most simplest form

$y = ax^b \qquad \frac{dy}{dx} = b \cdot a \cdot x^{b-1}$

Example 1: Differentiate the function $y = 2x^4$-3x-3.

$$\frac{dy}{dx} = 4 \cdot 2x^{4-1} - 1 \cdot 3x^{1-1} - 0 = 8x^3 - 3$$

When differentiating logarithms and exponents the following rules apply

$y = \ln x^b \qquad \frac{dy}{dx} = \frac{1}{x} \cdot b$

$y = e^{-x} \qquad \frac{dy}{dx} = -e^{-x}$

(Note that $y = \ln x^b = b \ln x$)

<u>Chain rule</u>

This is possibly the most important rule in differential calculus. Use the chain rule when you have to differentiate a function $\frac{dy}{dx} = f(g(x))$ that is itself a function of the variable that is being differentiated. Example functions are e, $\sqrt{\ }$ and $\ln$.

$\frac{df(g(x))}{dx} = \frac{d}{dx}(g(x)) \frac{d}{dx} g(x)$. The rule can also be expressed as $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ given that there has been a transformation of the function $u = g(x)$.

Example 2: Differentiate the following two functions using the Chain Rule.

$y = \sqrt{x^2 + 1}$ and $y = \ln(2x^3 + 4x)$

The solutions to the following equations are as follows:

let $u = x^2 + 1$

$\frac{du}{dx} = 2x$

$u = 2x^3 + 4x$

$\frac{du}{dx} = 6x^2 + 4$

Therefore as $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2} u^{-1/2} \cdot 2x$

$\frac{dy}{dx} = \frac{1}{u} \cdot \left(6x^2 + 4\right)$

Now back-transform such that

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}}$

$\frac{dy}{dx} = \frac{6x^2 + 4}{2x^3 + 4x} = \frac{2(3x^2 + 2)}{2(x^3 + 2x)} = \frac{3x^2 + 2}{x^3 + 2x}$

Product and quotient rules

Often the function to be differentiated has

$$\frac{d}{dx}\{f(x)g(x)\} = f(x)g'(x) + g(x)fg(x)$$

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example 3: Calculate $\frac{d}{dx}\{axe^{-bx}\}$ using the product rule.

$$\frac{d}{dx}\{axe^{-bx}\} = ax\frac{d}{dx}\{e^{-bx}\} + \frac{d}{dx}\{e^{-bx}\}\frac{d}{dx}\{ax\}$$

$$= axe^{-bx}\frac{d}{dx}\{-bx\} + e^{-bx} \times a$$

$$= -axbe^{-bx} + ae^{-bx}$$

$$= ae^{-bx}(1 - bx)$$

Example 4: Calculate $\frac{d}{dx}\left\{\frac{4x-2}{x^2+1}\right\}$ using the quotient rule.

$$\frac{d}{dx}\left\{\frac{4x-2}{x^2+1}\right\} = \frac{\left(x^2+1\right)\frac{d}{dx}\{4x-2\}-(4x-2)\frac{d}{dx}\{x^2+1\}}{\left(x^2+1\right)^2}$$

$$= \frac{\left(x^2+1\right)\times 4-(4x-2)\times 2x}{\left(x^2+1\right)^2}$$

$$= \frac{4x^2+4-8x^2+4x}{\left(x^2+1\right)^2}$$

$$\frac{-4x^2+4x+4}{\left(x^2+1\right)^2}$$

Examples

1. Differentiate the function $f(x)=10x^2+9x-4$

   $f'(x)=20x^2+9$

2. Differentiate the function $f(x)=\left(x^3-7\right)\left(2x^2+3\right)$

   $$\begin{aligned} f'(x) &= 3x^2\times\left(2x^2+3\right)+4x\times\left(x^3-7\right) \\ &= 6x^4+9x^2+4x^4-28x \\ &= 10x^4-19x^2-28x \end{aligned}$$

3. Differentiate the function $f(x)=\left(x^3-7\right)\left(2x^2+3\right)$

$$f'(w) = \frac{(w^3 - 7) \times 2 - 2w \times 3w^2}{(w^3 - 7)^2}$$
$$= \frac{2w^3 + 14 - 6w^3}{(w^3 - 7)^2}$$
$$= \frac{-4w^3 + 14}{(w^3 - 7)^2}$$

4. Differentiate the function $y = (3s)^{-4}$

$$f(y) = \frac{1}{81}s^{-4}$$
$$f'(y) = -\frac{5}{81}s^{-5}$$

5. Find $f'(x)$ if $y = x^x$

First, let $y = e^{x \ln x}$

$$f'(x) = e^{x \ln x} \times \frac{d}{x}\{x \ln x\}$$
$$= e^{x \ln x} \times \left(\frac{x}{x} + \ln x\right)$$
$$= -x \ln x(1 + \ln x)$$

6. Find $f'(x)$ if $f(x) = \ln\sqrt[3]{(2x+5)^2}$

Let $z = (2x+5)^2$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$$f(x) = \ln\left[(2x+5)^2\right]^{\frac{1}{3}} = \frac{1}{3}\ln(2x+5)^2 = \frac{2}{3}\ln(2x+5)$$

$$f'(x) = \frac{dy}{dz}\frac{dz}{dy} = \frac{2}{3}\times\frac{1}{2x+5}\times 2$$

$$= \frac{4}{6x+15}$$

7. Find $f'(x)$ if $f(x) = \ln\left\lfloor\sqrt{6x-1}(4x+5)^3\right\rfloor$ and $x > \frac{1}{6}$.

$$f(x) = \ln\left[(6x-1)^{\frac{1}{2}}(4x+5)^3\right]$$

$$= \ln(6x-1)^{\frac{1}{2}} + \ln(4x+5)^3$$

$$= \frac{1}{2}\ln(6x-1) + 3\ln(4x+5)$$

$$f'(x) = \frac{1}{2}\times\frac{1}{(6x-1)}\times 6 + 3\times\frac{1}{(4x+5)}\times 4$$

$$= \frac{3}{(6x-1)} + \frac{12}{(4x+5)}$$

$$= \frac{3(4x+5)+12(6x-1)}{(6x-1)(4x-5)}$$

$$= \frac{12x+15+72x-12}{(6x-1)(4x-5)} = \frac{84x-3}{(6x-1)(4x-5)}$$

8. Find $\frac{dy}{dx}$ if $y = e^{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = e^{\sqrt{x^2+1}} \times \frac{d}{dx}\left\{\sqrt{x^2+1}\right\} \times \frac{d}{dx}\left\{x^2\right\}$$

$$= e^{\sqrt{x^2+1}} \times \left(\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\right) \times 2x$$

$$= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

Integration

Integration is the opposite of differentiation. In calculus there is a method used to back-calculate the derivative of a function to its original form. If $f'(x) = f(x)/dx$ then the antiderivative of $f'(x)$ is $f(x)$.

In its simplest form

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$. It is crucial to always have the dx and the integration constant C in your expression and solution.

We can check our solution by taking the derivative

$\frac{d}{dx}\left\{\frac{x^{n+1}}{n+1} + C\right\} = \frac{n+1}{n+1} x^{(n+1)-1} + 0 = x^n$.

Integration, as with differentiation, has the following rules:

If we wish to integrate between two values say between some lower, L, and upper, U, bounds then the "Fundamental Theorem of the Calculus" applies. It is defined as

$$\int_L^U f'(x)dx = f(x)]_L^U = f(U) - f(L)$$

The Fundamental Theorem of the Calculus is also used to estimate the area under a curve or a volume if at least two-dimensions are integrated.

Example 5: Calculate $\int 2\text{x}dx$ between the bounds 1 and 3.

$$\int_1^3 \text{x}^2 dx = \left(\frac{x^3}{3}\right)\Bigg]_1^3 = \left(\frac{3^3}{3}\right) - \left(\frac{1^3}{3}\right) = 9 - \frac{1}{3} = \frac{27-1}{3} = \frac{26}{3}$$

Examples

9. $\int a^x dx = \left(\frac{1}{\ln a}\right)a^x + C$

10. $\int 4x^5 dx = \frac{4}{6}x^6 + C$

11. $\int \sqrt[3]{x^2}dx = \int x^{\frac{2}{3}}dx = \frac{x^{\frac{2}{3}+1}}{\left(\frac{2}{3}+1\right)} + C = \left(\frac{3}{5}\right)x^{\frac{5}{3}} + C$

12. $\int \frac{1}{x^2}dx = \int x^{-2}dx = -x^{-1} + C = -\frac{1}{x} + C$

13. Integrate $\int_{-2}^{3}\left(6x^2 - 5\right)dx$

$$\int_{-2}^{3}\left(6x^2 - 5\right)dx$$
$$= \left[2x^3 - 5x\right]_{-2}^{3}$$
$$= \left(2\times 3^3 - 5\times 3\right) - \left(2\times -2^3 - 5\times -2\right)$$
$$= 54 - 15 + 16 - 10$$
$$= 45$$

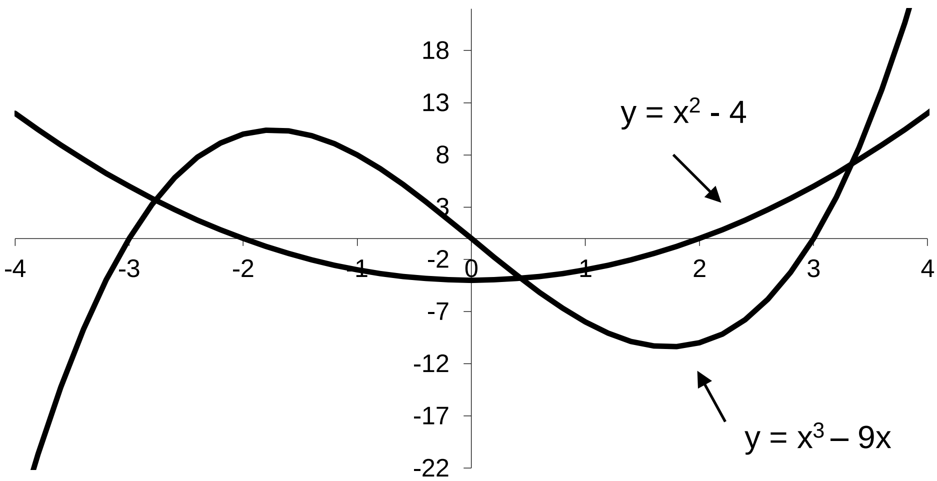
3. Finding the maxima/minima of a function

The maximum or minimum of a function can be found suing differentiation. For example, if we have a continuous function *f(y)* the maxima/minima, within range of *x* values *aka* the domain, are estimated easily by differentiating *f(y)* with respect to *x*, setting the equation to zero and then solving for *x*. These *x* values are then replaced into the original equations and solved for the corresponding *y* values. If there is more than one maximum or minimum value – in the case of a polynomial - then *x* would have more than one solution.

In the case of a high school solution if we have a quadratic of the form $y = ax^2 + bx + c$ then the maximum/minimum value is

$$\frac{dy}{dx} = 2ax + b = 0 \text{ such that } x = -\frac{b}{2a}$$

Example 6: A quadratic $y = x^2 - 4$ and polynomial $y = x^3 - 9$ are presented in the graph below. What are the maxima/ minima?



The maximum and minimum *x* values are therefore calculated by taking the derivative of *y* with respect to *x*, setting the derivative equal to zero and solving for *x*. The maximum and minimum *y* estimates are calculated by substituting the maximum and minimum *x* estimates back into the original equation.

For the quadratic the minimum over the interval [-2,2] is

$$\frac{dy}{dx} = 2x = 0$$

$x = 0$ and $y = -4$

and for the polynomial over the interval [-3,3]

$$\frac{dy}{dx} = 3x^2 - 9 = x^2 - 3 = 0$$

$x = \pm\sqrt{3}$ and $y \sim \pm 10.39$

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4. Finding the roots of a function

The root(s) of a function are those values where the function intersects the x-axis. Certain functions have roots that are easy to calculate and these are obtained by setting $f(x) = 0$ and solving for x. However, in non-linear and this is not a trivial task. A useful methods is that proposed by Newton and Raphson and is known as the "Newton-Raphson's methods of Roots" or NRMR.

NRMR is exact for linear problems and approximate for non-linear problems and based on the assumption that the solution to a function $f(x)$ can be estimated by a Taylor's series expansion around a reasonable estimate *x* close to the solution, say x_0. A Taylor's series expansion is of the form $f(x_1) = f(x_0) + (x_1 - x_0)f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) + \ldots$. In the NRMR only the linear terms are considered such that the solution x_1 can be found from an estimate x_0. Iterating with the new solution will solve the problem quickly.

A Taylor expansion around the initial estimate, x_0, will yield the approximate solution x_0 such that $f(x_1) \cong f(x_0) + (x_1 - x_0)f'(x_0)$. As we wish to solve for the function $f(x_1) = 0$ then $x_1 \cong x_0 - \frac{f(x_0)}{f'(x_0)}$ and is the general form of $x_1 \cong x_0 - \varepsilon$ where ε is some correction factor. On well-behaved function this method converges to the solution quadratically, and with linear functions only one iteration will be necessary.

NRMR can be depicted graphically for a function that has a first derivative that is itself a linear function. If we evaluate the derivative at our first estimate, x_0, then we can calculate the slope of the straight line that will intersect the objective function as a tangent. The slope, or the derivative, is calculated as $f'(x) = \frac{f(x_0) - 0}{x_0 - x_1}$. We can solve for x_1 such that $x_1 = x_0 - \frac{f(x_0)}{f'(x)}$, the solution from the Taylor series derivation. This derivation can be graphically illustrated in Figure 1.

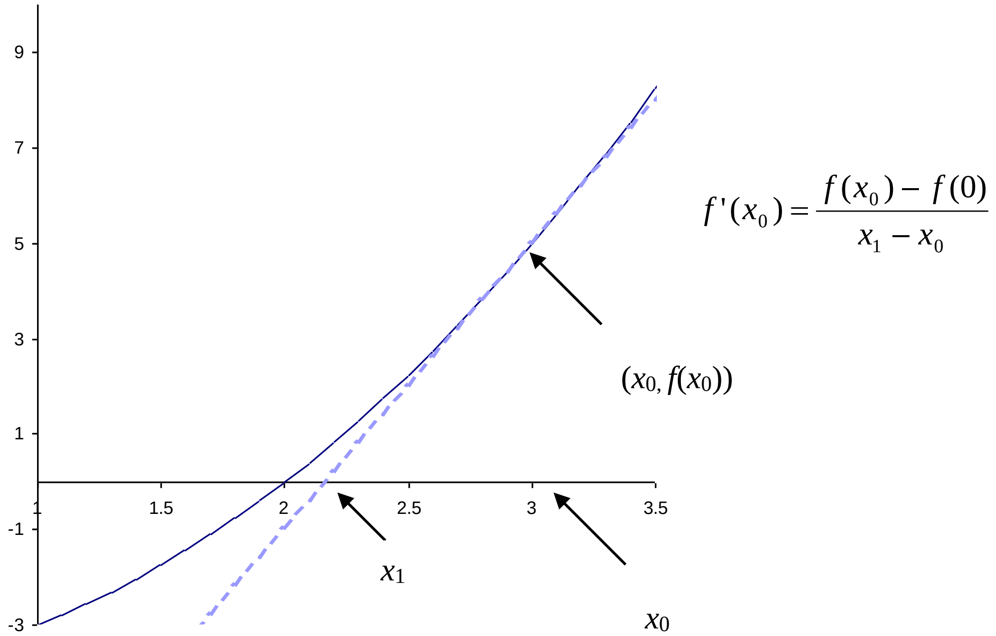

Figure 1: The Newton-Raphson method of roots. The initial guess, x_0, is used to calculate a (temporary) solution, x_1, through the calculation of the slope of linear derivative.

Example 7: What are the roots of Example 6 by a) factorizing and b) using Newton-Raphson's method?

For the quadratic the roots are $y = x^2 - 4 = (x-2)(x+2)$ such that x = 2 or x = -2, and $y = x^3 - 9x = x(x-3)(x+3)$ x = 3 or x = 0 or x = -3 for the polynomial.
We note that for the quadratic the minimum is found at (0,-4) therefore the roots will on either side of the minima because the quadratic is symmetric. The minimum root is found by setting an initial guess of $x = 3$.

The functions are $f(x) = x^2 - 4$ and $f'(x) = 2x$. Therefore $f(-3) = 5$ and $f'(-3) = -6$, so our first estimate is $x_1 \cong -3 - \frac{5}{-6} = \frac{-18+5}{6} = -\frac{13}{6}$. This is then resubstituted in such that

$$x_2 \cong -\frac{13}{6} - \frac{\left(\frac{169}{36} - 4\right)}{\left(-\frac{26}{12}\right)} = -\frac{13}{6} + \frac{12}{26}\left(\frac{169-144}{36}\right) = -\frac{13}{6} + \frac{12}{26}\left(\frac{25}{36}\right) = -2.006$$

After two more iterations a root of –2 is found. Similarly the rightmost root is found at 2. Intuition: as the minima was (0,-4) the roots will be symmetric around 0.

For the polynomial we noticed that the maxima/minima were found at $(\pm\sqrt{3}, \pm 10.23)$. The initial guesses are there for at $-\sqrt{3} - 1$, $\sqrt{3} + 1$ and the mid-point at 0. The solutions are as follows.

x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	εi
-2.732	4.196	13.392	-3.045	0.313
-3.045	-0.835	18.823	-3.000	-0.044
-3.000	-0.018	18.018	-3.000	-0.001
-3.000	-8.9E-06	18.000	-3.000	-4.94E-07

x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	εi
2.732	-4.196	13.392	3.045	-0.313
3.045	0.835	18.823	3.000	0.044
3.000	0.018	18.018	3.000	0.001
3.000	8.9E-06	18.000	3.000	4.9E-07

x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	εi
1	-8	-6.000	0.333	-1.333
-1.333	2.963	-8.667	0.008	-0.342
0.008	-0.077	-8.999	0.000	0.008

The roots are therefore found at $x = \pm 3$ and 0.