Black-hole initial data in the Einstein-Maxwell-Dilaton theory and prospects for numerical evolution

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Overview

- EMD gravity
- ADM formalism
- Vacuum case
- Electric field case
- Electric EMD solutions
- Dyonic EMD solutions
- GBSSN formalism
- Numerical implementation for vacuum
- Discussion

Einstein-Maxwell-Dilaton (EMD) gravity

- The action for the dilaton a-model is given by

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[R - 2\left(\partial\phi\right)^2 - \mathrm{e}^{-2a\phi}F^2\right] . \qquad (1)$$

Garfinkle, Horowitz, Strominger (1991)

- The metric equations of motion are

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi\,T_{\mu\nu}\,, \qquad (2)$$

with

$$T_{\mu\nu} = \frac{1}{4\pi}\left\{\partial_\mu\phi\partial_\nu\phi + \mathrm{e}^{-2a\phi}F_\mu{}^\alpha F_{\nu\alpha} - \tfrac{1}{4}g_{\mu\nu}\left[2\left(\partial\phi\right)^2 + \mathrm{e}^{-2a\phi}F^2\right]\right\} . \qquad (3)$$

- We also have the equations of motion for the matter fields

$$\Box\phi = -\frac{a}{2}\mathrm{e}^{-2a\phi}F^2\,, \qquad \nabla_\mu\left(\mathrm{e}^{-2a\phi}F^{\mu\nu}\right) = 0\,. \qquad (4)$$

$3+1$ decomposition

- Let n^μ be the timelike unit normal vector field to the spacelike hypersurface Σ. The spatial metric and extrinsic curvature are

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \,, \qquad K_{\mu\nu} = -\tfrac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu} \,. \tag{5}$$

- One can write the Gauss, Codazzi, and Ricci equations as

$$\mathcal{R}_{\mu\nu\lambda}{}^{\alpha} = -K_{\lambda\mu}K_{\nu}{}^{\alpha} + K_{\lambda\nu}K_{\mu}{}^{\alpha} + \gamma_{\lambda}{}^{\tau}\gamma_{\mu}{}^{\rho}\gamma_{\nu}{}^{\sigma}R_{\rho\sigma\tau}{}^{\beta}\gamma_{\beta}{}^{\alpha} \,, \tag{6}$$

$$\mathcal{D}_\nu K_{\mu\lambda} - \mathcal{D}_\mu K_{\nu\lambda} = -\gamma_{\lambda}{}^{\alpha}\gamma_{\mu}{}^{\beta}\gamma_{\nu}{}^{\sigma}R_{\sigma\beta\alpha}{}^{\rho}n_\rho \,, \tag{7}$$

$$\mathcal{L}_{\boldsymbol{n}}K_{\mu\nu} = -\alpha^{-1}\mathcal{D}_\mu\mathcal{D}_\nu\alpha - K_{\mu}{}^{\alpha}K_{\nu\alpha} - \gamma_{\mu}{}^{\sigma}\gamma_{\nu}{}^{\lambda}R^{\alpha}{}_{\sigma\lambda}{}^{\beta}n_\alpha n_\beta \,. \tag{8}$$

Baumgarte, Shapiro (2003)

ADM formalism

- Using the Gauss, Codazzi, and Ricci equations together with the field equations and

$$\rho := n^\mu n^\nu T_{\mu\nu}\,, \quad j^\mu := -\gamma^{\mu\alpha} n^\nu T_{\alpha\nu}\,, \quad S_{\mu\nu} := \gamma_\mu{}^\alpha \gamma_\nu{}^\beta T_{\alpha\beta}\,. \tag{9}$$

one arrives at the constraint equations

$$\mathcal{H} := \mathcal{R} + K^2 - K_{ij}K^{ij} - 16\pi\rho = 0\,, \tag{10}$$

$$\mathcal{M}_i := \mathcal{D}_j K^j{}_i - \mathcal{D}_i K - 8\pi j_i = 0\,, \tag{11}$$

and the evolution equation with $t^\mu := \alpha n^\mu + \beta^\mu$

$$\begin{aligned}\partial_t K_{ij} = &-\mathcal{D}_i\mathcal{D}_j\alpha + \alpha\left(\mathcal{R}_{ij} - 2K_i{}^\ell K_{j\ell} + K_{ij}K\right)\\ &- 8\pi\alpha\left[S_{ij} - \tfrac{1}{2}\gamma_{ij}\left(S-\rho\right)\right] + \beta^\ell\mathcal{D}_\ell K_{ij} + 2K_{\ell(i}\mathcal{D}_{j)}\beta^\ell\,. \end{aligned} \tag{12}$$

Arnowitt, Deser, Misner (2008)

Baumgarte, Shapiro (2003)

Initial value problem

- Consider time-symmetric initial data and a spatial metric with line-element

$$\mathrm{d}s^2 = \Phi^4 \mathrm{d}s_\flat^2 \,. \tag{13}$$

- The Hamiltonian constraint gives us

$$\delta^{ij}\mathcal{R}_{ij} = 2\delta^{ij}\left(E_i D_j + B_i M_j + \partial_i\phi\partial_j\phi\right) , \tag{14}$$

where $D_i = \mathrm{e}^{-2a\phi}E_i$, $M_i = \mathrm{e}^{-2a\phi}B_i$, and

$$\delta^{ij}\mathcal{R}_{ij} = -8\Phi^{-1}\Delta\Phi \,. \tag{15}$$

- We also have the following for the electric and magnetic fields

$$\delta^{ij}\partial_i\left(\Phi^2 D_j\right) = 0 \,, \qquad \delta^{ij}\partial_i\left(\Phi^2 B_j\right) = 0 \,. \tag{16}$$

UKBV, Ortín (2025)

Vacuum case

- For the case of a vacuum, the Hamiltonian constraint gives us

$$\Delta\Phi = 0\,, \tag{17}$$

which has the solution

$$\Phi = 1 + \sum_{i=1}^{N} \frac{\alpha_i}{|\boldsymbol{r} - \boldsymbol{c}_i|}\,. \tag{18}$$

- It is clear that $\Phi \to 1$ as $r \to \infty$, which corresponds to an assymptotically-flat region. One can define

$$A_j := 1 + \sum_{i \neq j} \frac{\alpha_i}{|\boldsymbol{c}_j - \boldsymbol{c}_i|}\,. \tag{19}$$

- We consider the limit as $r_j := |\boldsymbol{r} - \boldsymbol{c}_j| \to 0$, and write

$$\Phi = \frac{\alpha_j}{r_j}\left(1 + \frac{r_j A_j}{\alpha_j}\right) + \mathcal{O}\left(r_j\right)\,. \tag{20}$$

Brill, Lindquist (1963)

Vacuum case

- The line-element takes the form

$$ds^2 \sim \frac{\alpha_j^4}{r_j^4}\left(1+\frac{r_j A_j}{\alpha_j}\right)^4 ds_\flat^2\,. \tag{21}$$

- If we perform the coordinate transformation $r_j' = \alpha_j^2/r_j$, then

$$ds^2 \sim \left(1+\frac{\alpha_j A_j}{r_j'}\right)^4 ds_\flat^2\,. \tag{22}$$

  Then, as $r_j \to 0$ ($r_j' \to \infty$), $ds^2 \to ds_\flat^2$.
- The mass can be extracted via the fall-off:

$$ds^2 = \left(1+\frac{2m}{r}+\dots\right) ds_\flat^2\,. \tag{23}$$

Vacuum case

- We can extract the mass of each individual sheet

$$m_i = 2\alpha_i A_i \,, \tag{24}$$

for $i \in \{1, \ldots, N\}$, while in the $(N+1)$th sheet:

$$m_{N+1} = 2 \sum_{n=1}^{N} \alpha_i \,. \tag{25}$$

- The *interaction energy* is the difference:

$$m_{\text{int}} := m_{N+1} - \sum_{i=1}^{N} m_i \,. \tag{26}$$

Brill, Lindquist (1963)

- In the case of a single centre $(N = 1)$, we have $m_{\text{int}} = 0$ and this corresponds to the time-symmetric slice of the Schwarzschild solution.

Electric field

- We consider the following ansatz where $\Delta\chi = \Delta\psi = 0$:

$$\mathrm{d}s^2 = \chi^2\psi^2\mathrm{d}s_\flat^2\,. \tag{27}$$

- In such a case, the Hamiltonian constraint gives the following

$$E_i = \partial_i \ln\left(\chi/\psi\right)\,. \tag{28}$$

- As was done for the Schwarzschild case, we perform the expansion

$$\chi = 1 + \sum_{n=1}^{N} \frac{\alpha_i}{|\boldsymbol{r} - \boldsymbol{c}_i|}\,, \tag{29}$$

$$\psi = 1 + \sum_{n=1}^{N} \frac{\beta_i}{|\boldsymbol{r} - \boldsymbol{c}_i|}\,, \tag{30}$$

- There is an asymptotically-flat region as $r \to \infty$.

Brill, Lindquist (1963)

Electric field

- We define

$$A_j \,, B_j := 1 + \sum_{i \neq j} \frac{\alpha_i \,, \beta_i}{|\boldsymbol{c}_j - \boldsymbol{c}_i|} \,. \tag{31}$$

- In the limit as $r_j \to 0$,

$$\mathrm{d}s^2 \sim \frac{\alpha_j^2 \beta_j^2}{r_j^4} \left(1 + \frac{r_j A_j}{\alpha_j}\right)^2 \left(1 + \frac{r_j B_j}{\beta_j}\right)^2 \mathrm{d}s_\flat^2 \,. \tag{32}$$

- One can perform the coordinate transformation: $r_j' := \alpha_j \beta_j / r_j$, giving

$$\mathrm{d}s^2 \sim \left(1 + \frac{\beta_j A_j}{r_j'}\right)^2 \left(1 + \frac{\alpha_j B_j}{r_j'}\right)^2 \mathrm{d}s_\flat^2 \,. \tag{33}$$

Brill, Lindquist (1963)

Electric field

- The mass in the $i \in \{1, \ldots, N\}$ sheet is

$$m_i = \beta_i A_i + \alpha_i B_i \,, \tag{34}$$

while in the $(N+1)$th sheet we have

$$m_{N+1} = \sum_{n=1}^{N} (\alpha_i + \beta_i) \,. \tag{35}$$

- For the electric charge in each sheet, we use $\frac{1}{4\pi} \int \mathrm{d}A E_i n^i$

$$q_i = -(B_i \alpha_i - A_i \beta_i) \,, \qquad q_{N+1} = \sum_{i=1}^{N} (\alpha_i - \beta_i) \,, \tag{36}$$

and we have $\sum_i q_i = -q_{N+1}$.

Brill, Lindquist (1963)

Electric field

- The interaction energy can be written as

$$m_{\text{int}} = -\sum_{i=1}^{N} \sum_{j \neq i} \frac{\alpha_i \beta_j + \alpha_j \beta_i}{r_{ij}}, \tag{37}$$

where $r_{ij} := |\boldsymbol{c}_i - \boldsymbol{c}_j|$. It is clear that $m_{\text{int}} = 0$ when $N = 1$.

- For centres that have large separations:

$$m_{\text{int}} \approx -\sum_{i<j} \frac{m_i m_j - q_i q_j}{r_{ij}}. \tag{38}$$

For $N = 2$, we can recognise an attractive gravitational potential, and a repulsive (for like charges) Coulombian potential.

Brill, Lindquist (1963)

Electric and dilaton fields

- Consider the line-element:

$$ds^2 = (UV)^{\frac{2}{1+a^2}}\; W^{\frac{4a^2}{1+a^2}} ds_\flat^2 \,, \tag{39}$$

  for harmonic functions

$$U, V, W = 1 + \sum_{n=1}^{N} \frac{\alpha_i \,, \beta_i \,, \sigma_i}{|\boldsymbol{r} - \boldsymbol{c}_i|} \,. \tag{40}$$

- The electric field is given by

$$D_i = \frac{e^{-a\phi_\infty}}{\sqrt{1+a^2}} \left(\frac{UV}{W^2}\right)^{\frac{a^2}{1+a^2}} \partial_i \ln\left(\frac{U}{V}\right) \,, \tag{41}$$

  while the dilaton field is given by

$$e^{-a\phi} = e^{-a\phi_\infty} \left(\frac{UV}{W^2}\right)^{\frac{a^2}{1+a^2}} \tag{42}$$

Cvetic, Gibbons, Pope (2015)

Electric and dilaton fields

- The mass in the $(N+1)$th sheet:

$$m_{N+1} = \sum_{i=1}^{N} \frac{\alpha_i + \beta_i + 2a^2\sigma_i}{1+a^2} \,, \tag{43}$$

while in the ith sheet it is

$$m_i = \frac{\left(\alpha_i \beta_i \sigma_i^{2a^2}\right)^{\frac{1}{1+a^2}}}{1+a^2} \left(\frac{P_i}{\alpha_i} + \frac{Q_i}{\beta_i} + \frac{2a^2 S_i}{\sigma_i} \right) \,, \tag{44}$$

where

$$P_i \,, Q_i \,, S_i := 1 + \sum_{i \neq j} \frac{\alpha_i \,, \beta_i \,, \sigma_i}{r_{ij}} \,. \tag{45}$$

- The electric charge is

$$q_i = \frac{\mathrm{e}^{-a\phi_\infty}}{\sqrt{1+a^2}} \left(P_i \beta_i - Q_i \alpha_i \right) \,. \tag{46}$$

Electric and dilaton fields

- We extract the scalar charge via the fall-off:

$$\phi \sim \phi_\infty + \frac{\Sigma}{r}\,, \tag{47}$$

and obtain

$$\Sigma_{N+1} = \frac{a}{1+a^2}\sum_{i=1}^{N}\left(2\sigma_i - \alpha_i - \beta_i\right)\,, \tag{48}$$

$$\Sigma_i = \frac{a\left(\alpha_i\beta_i\sigma_i^{2a^2}\right)^{\frac{1}{1+a^2}}}{1+a^2}\left[\frac{2S_i}{\sigma_i} - \frac{P_i}{\alpha_i} - \frac{Q_i}{\beta_i}\right]\,. \tag{49}$$

UKBV, Ortín (2025)

Electric and dilaton fields

- Consider the case of a single ($N = 1$) black-hole

$$m_2 = \frac{\alpha_1 + \beta_1 + 2a^2\sigma_1}{1 + a^2} , \tag{50}$$

$$m_1 = \frac{\left(\alpha_1\beta_1\sigma_1^{2a^2}\right)^{\frac{1}{1+a^2}}}{1 + a^2} \left(\frac{1}{\alpha_1} + \frac{1}{\beta_1} + \frac{2a^2}{\sigma_1}\right) , \tag{51}$$

$$\Sigma_2 = \frac{a}{1 + a^2} \left(2\sigma_1 - \alpha_1 - \beta_1\right) , \tag{52}$$

$$\Sigma_1 = \frac{a\left(\alpha_1\beta_1\sigma_1^{2a^2}\right)^{\frac{1}{1+a^2}}}{1 + a^2} \left(\frac{2}{\sigma_1} - \frac{1}{\alpha_1} - \frac{1}{\beta_1}\right) , \tag{53}$$

$$\phi_1 = \phi_\infty + \frac{a}{1 + a^2} \ln\left(\frac{\sigma_1^2}{\alpha_1\beta_1}\right) . \tag{54}$$

Electric and dilaton fields

- We notice that the dilaton field takes on the same value in the two assymptotic regions when $\sigma_1^2 = \alpha_1\beta_1$. In addition, $m_1 = m_2$, and thus $m_{\text{int}} = 0$. Furthermore, $\Sigma_1 = \Sigma_2$.
- Such initial data correspond to well-known static EMD black-hole solutions (see **Cvetic, Gibbons, Pope (2015)**); another example of vanishing self-interaction energy corresponding to stationay solutions.
- Given the constraint on the integration parameters, there are now only two integration parameters; corresponding to two independent physical constants and there is no primary scalar hair.

Electric, magnetic, and dilaton fields

- We consider the following line-element containing four harmonic functions

$$\mathrm{d}s^2 = CDST\mathrm{d}s_\flat^2\,, \tag{55}$$

$$C\,,D\,,S\,,T = 1+\sum_{i=1}^{N}\frac{\sigma_i\,,\rho_i\,,\lambda_i\,,\tau_i}{|\boldsymbol{r}-\boldsymbol{c}_i|}\,. \tag{56}$$

- For the electric and magnetic fields, we have

$$D_i = \alpha\frac{e^{-\phi_\infty}}{\sqrt{2}}\sqrt{\frac{CS}{DT}}\partial_i \ln\left(\frac{C}{S}\right)\,, \tag{57}$$

$$B_i = \beta\frac{e^{\phi_\infty}}{\sqrt{2}}\sqrt{\frac{DT}{CS}}\partial_i \ln\left(\frac{D}{T}\right)\,, \tag{58}$$

where $\alpha^2=\beta^2=1$ and $\phi=\phi_\infty-\frac{1}{2}\ln\left(\frac{CS}{DT}\right)$.

UKBV, Ortín (2025)

Electric, magnetic, and dilaton fields

- We use

$$\tilde{C}_i\,,\tilde{D}_i\,,\tilde{S}_i\,,\,\tilde{T}_i := 1 + \sum_{j\neq i} \frac{\sigma_j\,,\rho_j,\,,\lambda_j\,,\tau_j}{r_{ij}}\,. \tag{59}$$

- For the electric and magnetic charges, we find

$$q_i = \alpha\frac{\mathrm{e}^{-\phi_\infty}}{\sqrt{2}} \sum_{i=1}^{N} \left(\tilde{C}_i\lambda_i - \tilde{S}_i\sigma_i\right)\,, \tag{60}$$

$$p_i = \beta\frac{\mathrm{e}^{\phi_\infty}}{\sqrt{2}} \left(\tilde{D}_i\tau_i - \tilde{T}_i\rho_i\right)\,, \tag{61}$$

which satisfy $\sum_i q_i = -q_{N+1}$ and $\sum_i p_i = -p_{N+1}$.

Electric, magnetic, and dilaton fields

- For the mass and scalar charge, we find

$$m_{N+1} = \tfrac{1}{2} \sum_{n=1}^{N} (\sigma_i + \rho_i + \lambda_i + \tau_i) \ , \tag{62}$$

$$\begin{aligned} m_i &= \frac{\tilde{C}_i}{2} \left(\frac{\rho_i \lambda_i \tau_i}{\sigma_i} \right)^{1/2} + \frac{\tilde{D}_i}{2} \left(\frac{\sigma_i \lambda_i \tau_i}{\rho_i} \right)^{1/2} \\ &+ \frac{\tilde{S}_i}{2} \left(\frac{\rho_i \sigma_i \tau_i}{\lambda_i} \right)^{1/2} + \frac{\tilde{T}_i}{2} \left(\frac{\rho_i \lambda_i \sigma_i}{\tau_i} \right)^{1/2} \ , \end{aligned} \tag{63}$$

$$\Sigma_{N+1} = \tfrac{1}{2} \sum_{i=1}^{N} (\rho_i + \tau_i - \sigma_i - \lambda_i) \ , \tag{64}$$

$$\begin{aligned} \Sigma_i &= -\left[\frac{\tilde{C}_i}{2} \left(\frac{\rho_i \lambda_i \tau_i}{\sigma_i} \right)^{1/2} - \frac{\tilde{D}_i}{2} \left(\frac{\sigma_i \lambda_i \tau_i}{\rho_i} \right)^{1/2} \right. \\ &\left. + \frac{\tilde{S}_i}{2} \left(\frac{\rho_i \sigma_i \tau_i}{\lambda_i} \right)^{1/2} - \frac{\tilde{T}_i}{2} \left(\frac{\rho_i \lambda_i \sigma_i}{\tau_i} \right)^{1/2} \right], \end{aligned} \tag{65}$$

Eletric, magnetic, and dilaton fields

- Let us now consider the $N = 1$ case, we have $\phi_1 = \phi_\infty - \ln(\frac{\sigma_1\lambda_1}{\rho_1\tau_1})$ and

$$m_2 = \frac{1}{2}\left(\sigma_1 + \rho_1 + \lambda_1 + \tau_1\right) , \tag{66}$$

$$m_1 = \frac{1}{2}\left[\left(\frac{\rho_1\lambda_1\tau_1}{\sigma_1}\right)^{1/2} + \left(\frac{\sigma_1\lambda_1\tau_1}{\rho_1}\right)^{1/2} + \left(\frac{\rho_1\sigma_1\tau_1}{\lambda_1}\right)^{1/2} + \left(\frac{\rho_1\lambda_1\sigma_1}{\tau_1}\right)^{1/2} \right], \tag{67}$$

$$\Sigma_2 = \tfrac{1}{2}\left(\rho_1 + \tau_1 - \sigma_1 - \lambda_1\right) , \tag{68}$$

$$\Sigma_1 = -\frac{1}{2}\left[\left(\frac{\rho_1\lambda_1\tau_1}{\sigma_1}\right)^{1/2} - \left(\frac{\sigma_1\lambda_1\tau_1}{\rho_1}\right)^{1/2} + \left(\frac{\rho_1\sigma_1\tau_1}{\lambda_1}\right)^{1/2} - \left(\frac{\rho_1\lambda_1\sigma_1}{\tau_1}\right)^{1/2} \right], \tag{69}$$

Electric, magnetic, and dilaton fields

- In the case where $\sigma_1 \lambda_1 = \rho_1 \tau_1$, we have $\phi_\infty = \phi_0$, $\Sigma_1 = \Sigma_2$, and $m_{\text{int}} = 0$.
- These give us constant time slices of the static black-hole solutions constructed in **Gibbons (1982)** and **Gibbons, Maeda (1988).**
- This is another example of vanishing self-interaction energy corresponding to the absence of primary scalar hair, and coinciding with stationary black-hole solutions.

GBSSN formalism

- The Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formalism is a strongly-hyperbolic modification of the ADM formalism, involving a conformal rescaling of the spatial metric, a decomposition of the extrinsic curvature into its trace and trace-free parts, and the use of additional auxiliary variables.

  Nakamura, Oohara, Kojima (1987)

  Shibata, Nakamura (1995)

  Baumgarte, Shapiro (1998)
- The standard BSSN formalism involves a conformal spatial metric with a determinant of unity, which is not immediately suitable for spherical polar coordinates.
- The generalised-BSSN (GBSSN) formalism removes this assumption — generalising to arbitrary coordinates.

  Brown (2005)

  Brown (2008)

GBSSN formalism

- We perform a rescaling of the spatial metric

$$\bar{\gamma}_{ij} = \mathrm{e}^{-4\phi}\gamma_{ij} \,, \tag{70}$$

where ϕ is the *conformal factor* and is given by

$$\phi = \frac{1}{12}\ln\left(\gamma/\bar{\gamma}\right) \,. \tag{71}$$

- The determinant of the conformal factor evolves as:

$$\left(\partial_t - \mathcal{L}_{\boldsymbol{\beta}}\right)\bar{\gamma} = -2\bar{\gamma}\bar{\mathcal{D}}_i\beta^i \,. \tag{72}$$

while the conformal factor has the evolution equation

$$\partial_t\phi = -\frac{\alpha}{6}K + \frac{1}{6}\partial_i\beta^i + \frac{1}{6}\beta^i\partial_i\ln\sqrt{\bar{\gamma}} + \beta^i\partial_i\phi \,. \tag{73}$$

- The evolution of the conformal spatial metric is given by

$$\begin{aligned}\partial_t\bar{\gamma}_{ij} = -2\alpha\bar{A}_{ij} - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\beta^k\partial_k\ln\sqrt{\bar{\gamma}} + \beta^k\partial_k\bar{\gamma}_{ij} \\ + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{jk}\partial_i\beta^k \,.\end{aligned} \tag{74}$$

GBSSN formalism

- The extrinsic curvature is decomposed into a trace and trace-free part, and the latter rescaled

$$A_{ij} = K_{ij} - \tfrac{1}{3}K\gamma_{ij}\,, \qquad \bar{A}_{ij} = \mathrm{e}^{-4\phi}A_{ij}\,. \tag{75}$$

- The trace and trace-free parts have evolution equations

$$\partial_t K = -D^i D_i\alpha + \beta^i\partial_i K + 4\pi\alpha\left(\rho + S\right) + \alpha\left(\bar{A}_{ij}\bar{A}^{ij} + \tfrac{1}{3}K^2\right)\,. \tag{76}$$

$$\begin{aligned}\partial_t\bar{A}_{ij} = {}& \alpha K\bar{A}_{ij} - \tfrac{2}{3}\bar{A}_{ij}\partial_k\beta^k - \tfrac{2}{3}\bar{A}_{ij}\beta^k\partial_k\ln\sqrt{\bar{\gamma}} \\ & - \mathrm{e}^{-4\phi}\left(D_iD_j\alpha\right)^{\mathrm{TF}} - 2\alpha\bar{A}_i{}^k\bar{A}_{jk} + \alpha\mathrm{e}^{-4\phi}\mathcal{R}_{ij}^{\mathrm{TF}} \\ & + \beta^k\partial_k\bar{A}_{ij} + 2\bar{A}_{k(i}\partial_{j)}\beta^k - 8\pi\alpha e^{-4\phi}S_{ij}^{\mathrm{TF}},.\end{aligned} \tag{77}$$

- One also defines

$$\bar{\Gamma}^i := \bar{\gamma}^{jk}\bar{\Gamma}^i{}_{jk}\,. \tag{78}$$

GBSSN formalism

- The evolution equation for the conformal connection function is

$$\begin{aligned}\partial_t \bar{\Gamma}^i = {} & \tfrac{1}{3}\bar{\gamma}^{ik}\left(\partial_k\partial_j\beta^j + \partial_k\beta^j\partial_j \ln\sqrt{\bar{\gamma}} + \beta^j\partial_k\partial_j \ln\sqrt{\bar{\gamma}}\right) \\ & + 12\alpha\bar{A}^{ij}\partial_j\phi + 2\alpha\bar{\Gamma}^i{}_{jk}\bar{A}^{jk} - \tfrac{4}{3}\alpha\bar{\gamma}^{ij}\partial_j K - 2\bar{A}^{ij}\partial_j\alpha \\ & + \beta^j\partial_j\bar{\Gamma}^i + \bar{\gamma}^{jk}\partial_j\partial_k\beta^i + \tfrac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \tfrac{2}{3}\bar{\Gamma}^i\beta^j\partial_j \ln\sqrt{\bar{\gamma}} \\ & - \bar{\Gamma}^j\partial_j\beta^i - 16\pi\alpha\bar{\gamma}^{ij}j_j\,. \end{aligned} \tag{79}$$

- In the case where $\bar{\gamma} = 1$, the GBSSN formalism reduces to the standard BSSN formalism.
- One issue is that $\bar{\Gamma}^i$ are not components of a true vector. In fact, for a flat conformal spatial metric in spherical polar coordinates, we have $\bar{\Gamma}^r = -2/r$, which is singular.

GBSSN formalism

- We can define the following which are components of a true vector

$$\bar{\Delta}^i = \bar{\Gamma}^i - \bar{\gamma}^{jk}\, {}^{B}\Gamma^i{}_{jk} \,. \tag{80}$$

Brown (2009)

Alcubierre, Mendez (2011)

- We now assume spherical symmetry and introduce the evolution variables a, b, and A_a as follows:

$$\bar{\gamma}_{ij} = \mathrm{diag}\left(a, br^2, br^2 \sin^2\theta\right) \,, \tag{81}$$

$$\bar{A}_i{}^j = \mathrm{diag}\left(A_a, -A_a/2, -A_a/2\right) \,, \tag{82}$$

as well as

$$\bar{\Delta}^r = \bar{\Gamma}^r + \frac{2}{rb} = -\frac{\frac{\partial}{\partial r} b}{ab} + \frac{\frac{\partial}{\partial r} a}{2a^2} + \frac{2}{rb} - \frac{2}{ra} \,. \tag{83}$$

Alcubierre, Mendez (2011)

Numerical evolution

- For the evolution, we use non-advective $1 + \log$ slicing together with the Gamma-driver condition:

$$\partial_t \beta^r = B^r \,, \qquad \partial_t B^r = \tfrac{3}{4}\partial_t \bar{\Delta}^r \,. \tag{84}$$

Alcubierre, Mendez (2011)

- For our initial data, we take $a = b = 1$, and $\bar{\Delta}^r = K = A_a = \beta^r = B^r = 0$. We also use the evolution variable $\chi := \mathrm{e}^{-2\phi}$ and the initial data

$$\alpha = \chi = \left(1 + \frac{M}{2r}\right)^{-2} , \tag{85}$$

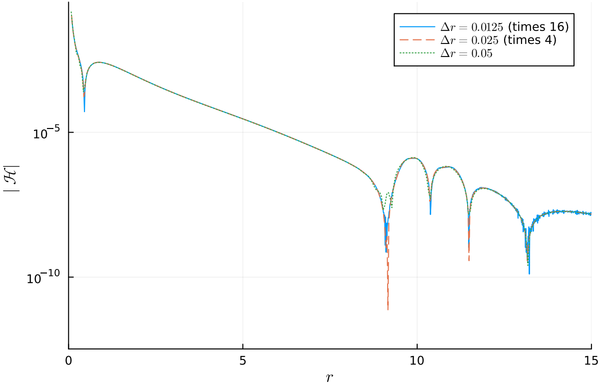
and we set $M = 1$.

- We use RK4 for evolution, and second-order finite differencing for spatial derivatives.

UKBV, Pollney, Dombriz (2023) (see for $\partial_t \bar{\Delta}^r$, $\partial_t A_a$, $\mathcal{H}$, etc.)

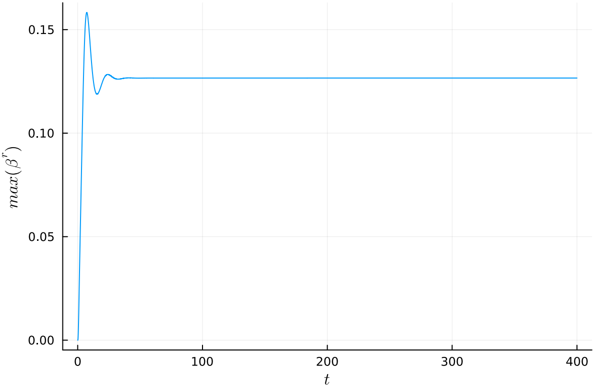
Numerical results

- Below we plot the Hamiltonian constraint at $t = 10$ with spatial resolutions of $\Delta r = 0.0125$, $\Delta r = 0.025$, and $\Delta r = 0.05$, rescaled according to second–order convergence. We have kept $\Delta t = \Delta r/2$.



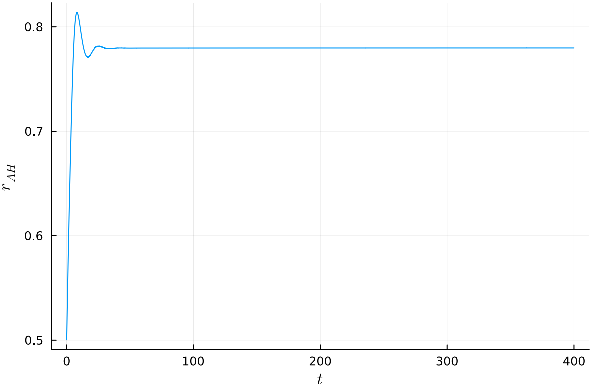
Numerical results

- We can compute the maximum value of the shift, β^r (using $\Delta r = 0.01$).



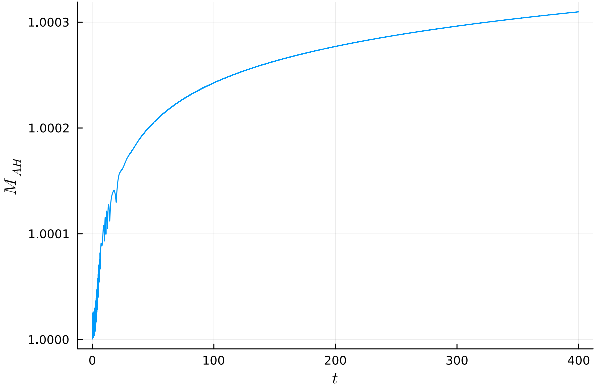
Numerical results

- We can also compute the horizon position:



Numerical results

- Across the evolution, we can compute the mass using $M_{AH} = \sqrt{\frac{A_{AH}}{16\pi}}$:



Discussion and future work

- In the vacuum and electric field cases, the self-interaction energy is zero.
- In the electric and dyonic EMD cases, the self-interaction energy can be non-zero and there is primary scalar hair.
- When setting the self-interaction energy to be zero, there is an absence of primary scalar hair.
- It is well-known that the GBSSN formalism can be used to evolve Schwarzschild initial data; the same formalism may be suitable to evolve the EMD initial data discussed here.
- Given some EMD initial data with non-vanishing self-interaction energy, would its evolution reach a stable configuration?
- If a stable state is reached, how would we verify the absence of primary scalar hair? One possibility could be to vary the scalar charge in the initial data and compute the scalar charge at the end.

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