Relativistic Wave Equations for the Photon & Graviton

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Nithecs GW Workshop

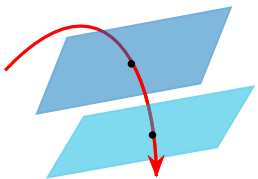
Introduction

Spatial-temporal Decomposition

- Spatial-temporal $(1 + 3)$ covariant formalism:

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu \tag{1.1}$$

- This decomposition enables projection of any tensor into parts parallel and orthogonal to a velocity u_μ.



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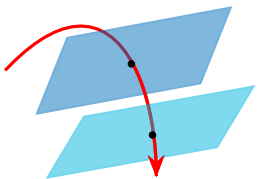

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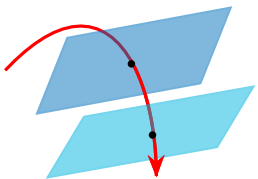

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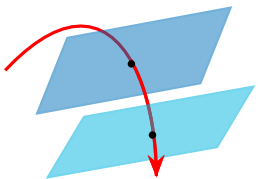

- Projected quantities require projected operators to act as three-dimensional operators:

$$\overline{\nabla}_\alpha S^\mu{}_\nu = h_\alpha{}^\sigma h^\mu{}_\tau h_\nu{}^\lambda \nabla_\sigma S^\tau{}_\lambda \tag{1.2}$$

$$\operatorname{curl} V_\mu = \epsilon_{\mu\nu\sigma}\overline{\nabla}^\nu V^\sigma \qquad \operatorname{curl} S_{\mu\nu} = \epsilon_{\alpha\sigma(\mu}\overline{\nabla}^\alpha S_{\nu)}{}^\sigma \tag{1.3}$$

$$\operatorname{div} V = \overline{\nabla}^\mu V_\mu \qquad (\operatorname{div} S)_\mu = \overline{\nabla}^\nu S_{\mu\nu} \tag{1.4}$$

- Projected, symmetric, trace free (PSTF) tensors are noted

$$V_{\langle\mu\rangle} = h_\mu{}^\nu V_\nu \tag{1.5}$$

$$S_{\langle\mu\nu\rangle} = \left(h_{(\mu}{}^\alpha h_{\nu)}{}^\sigma - \tfrac{1}{3} h_{\mu\nu} h^{\alpha\sigma}\right) S_{\alpha\sigma} \tag{1.6}$$

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Photon Wave Equation

Maxwell's Equations

In curved spacetime, Maxwell's equations are

$$\overline{\nabla}_\mu E^\mu = \rho - 2\omega_\mu B^\mu \tag{2.1}$$

$$\overline{\nabla}_\mu B^\mu = 2\omega_\mu E^\mu \tag{2.2}$$

$$\dot{E}_{\langle\mu\rangle} = \Xi_{\mu\nu} E^\nu + \epsilon_{\mu\nu\sigma}\dot{u}^\nu B^\sigma + \operatorname{curl} B_\mu - j_\mu \tag{2.3}$$

$$\dot{B}_{\langle\mu\rangle} = \Xi_{\mu\nu} B^\nu - \epsilon_{\mu\nu\sigma}\dot{u}^\nu E^\sigma - \operatorname{curl} B_\mu. \tag{2.4}$$

Kinematic quantities: $\Xi_{\mu\nu} = \sigma_{\mu\nu} + \epsilon_{\mu\nu\sigma}\omega^\sigma - \frac{2}{3}\Theta h_{\mu\nu}$.

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$\nabla \cdot E = \rho$

$\nabla \cdot B = 0$

$\partial_t B = \nabla \times E$

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Relativistic Photon Wave Equation

- Introducing the rank-1 Riemann-Silberstein (RS) tensor,

$$F_\mu = E_\mu + iB_\mu \tag{2.5}$$

- The *source-free* Maxwell's equations may be rewritten i.t.o. the RS-tensor,

$$\overline{\nabla}_\mu F^\mu = 2i\omega_\mu F^\mu \tag{2.6}$$

$$\dot{F}_\mu = -i\epsilon_{\mu\nu\sigma}(iF^\nu\omega^\sigma + F^\sigma\dot{u}^\nu + \overline{\nabla}^\nu F_\sigma) \tag{2.7}$$

$$i\dot{F}_\mu = i\zeta_\nu(-i\omega^\nu + \dot{u}^\nu + \overline{\nabla}^\nu)F_\mu \tag{2.8}$$

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$$i\dot{F}_\mu = i\zeta_\nu(-i\omega^\nu + \dot{u}^\nu + \overline{\overline{\nabla}}^\nu)F_\mu$$

- Schrödinger equation: $i\partial_t\psi = H\psi$.
- The right-hand side is the Hamiltonian.
- Constraint: terms contained within the Hamiltonian must be purely anti-symmetric.
- In Minkowski spacetime,

$$i\zeta_\nu D^\nu F_\mu \longrightarrow i\zeta_\nu \partial^\nu F_\mu = 0 \qquad (2.9)$$

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Anti-symmetry Constraint

- The Levi-Civita form may be expressed as a set of matrices — constructed from the Lorentz group: $M_i = L_i + K_i$

$$\epsilon_{\mu\nu\sigma} \equiv \eta_{ij}\zeta^j \tag{2.10}$$

- where the ζ-matrices encode spin-1,

$$\zeta^0 = I_4, \quad \zeta^1 = \gamma^0\gamma^1\gamma^3, \quad \zeta^2 = \gamma^0\gamma^5, \quad \zeta^3 = \gamma^3\gamma^5\gamma^1$$

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Properties

- Fulfils spinor formalism: 8×8 matrices encodes 4×4 block matrices (which are the ζ matrices).
- Fermionic correspondence: the photon only has two helicity states ($h = \pm 1$), since there is no rest state.
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Geometric Phase

- Dirac equation analogue: minimal coupling,

$$\partial_\mu \longrightarrow \partial_\mu + A_\mu \tag{2.11}$$

- This coupling produces a phase,

$$\phi = \oint \mathrm{d}\lambda \, A(\lambda) \tag{2.12}$$

- Choice of connection $\longrightarrow$ geometric phase.

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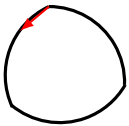

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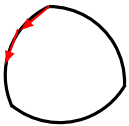

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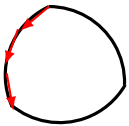

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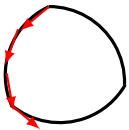

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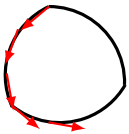

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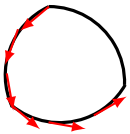

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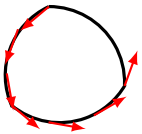

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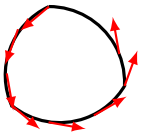

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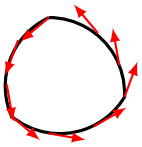

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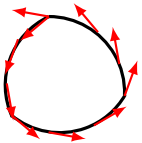

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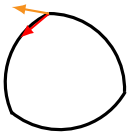

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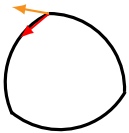

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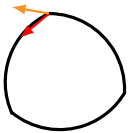

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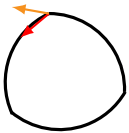

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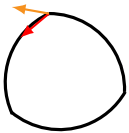

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Graviton Wave Equation

Weyl Tensor & The Bianchi Identities

- From the Bianchi identities, two rank-2 fields from the Weyl tensor are respectively defined as,

$$E_{\mu\nu} = C_{\mu\nu\alpha\tau} u^{\alpha} u^{\tau} \tag{3.1}$$

$$H_{\mu\nu} = \tfrac{1}{2} \epsilon_{\mu\alpha\tau} C^{\alpha\tau}{}_{\nu\sigma} u^{\sigma}. \tag{3.2}$$

- These define *electric*- and *magnetic*-analogue fields,

$$E_{\mu} = F_{\mu\nu} u^{\nu}$$

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Maxwell-analogue GW Equations

- We consider the *source-free* equations,

$$\overline{\nabla}^{\nu} E_{\nu\mu} = \epsilon_{\mu\nu\alpha}\sigma^{\nu}{}_{\sigma} H^{\sigma\alpha} - 3H_{\mu\nu}\omega^{\nu} \tag{3.3}$$

$$\overline{\nabla}^{b} H_{\nu\mu} = \epsilon_{\mu\nu\alpha}\sigma^{\nu}{}_{\sigma} E^{\alpha\sigma} + 3E_{\mu\nu}\omega^{\nu} \tag{3.4}$$

$$\dot{E}_{\mu\nu} = \operatorname{curl} H_{\mu\nu} + 2\dot{u}^{\alpha}\epsilon_{\alpha\sigma\mu}H_{\nu}{}^{\sigma} + \Theta E_{\mu\nu} + 3\sigma_{\alpha\langle\mu}E^{\alpha}{}_{\nu\rangle} - \omega_{\alpha\langle\mu}E^{\alpha}{}_{\nu\rangle} \tag{3.5}$$

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- Maxwell's equations (rank-1 RS)
- Lorentz group (4×4)
- Fermionic correspondence: $h = \pm 1$
- Spin-1 relativistic wave equation
- Geometric phase (minimal coupling: A_μ)

GW-dynamics

- GW analogue equations (rank-2 RS)
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- Likewise, h_+ & $h_\times$
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Electrodynamics	GW-dynamics
Maxwell's equations (rank-1 RS)	GW analogue equations (rank-2 RS)
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