Transformation optics and gravitational waves

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Electromagnetic fields and forces

- The electric and magnetic forces were unified by Maxwell (1873)
- Electromagnetic fields in vacuum obey

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- The Lorentz force law is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

EM fields in a medium

- In the presence of a *dielectric*, Maxwell's laws are modified

$$\nabla \cdot \mathbf{D} = \rho_f$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

- The free charge and current densities: ρ_{f} and $\mathbf{J}_{\mathrm{f}}$
- For *linear* media the constitutive relations are:

$$\mathbf{D} = \epsilon \mathbf{E}$$
$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

EM in Special Relativity

- Electromagnetism is a Lorentz-invariant theory
- This is made explicit by means of the field-strength (Faraday) tensor, $F^{\mu\nu}$
- In terms of the 4-potential: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- The electric and magnetic field components can be read off

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

- Maxwell's equations now read:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu$$

$$\partial_\mu [\tilde{F}^{\mu\nu}] = \partial_\mu \left[\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \right] = 0$$

- The electromagnetic field components above are valid in the laboratory frame where $u^\mu = (1,0,0,0)$
- We can define them in ANY inertial reference frame with relative velocity, u^μ
- Project tensors along and orthogonal to an observer via $h_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$
- The electric and magnetic fields are thus:

$$E^\mu = F^{\mu\nu} u_\nu$$

$$B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$$

- The Faraday tensor can be written as

$$F^{\mu\nu} = E^{\nu} u^{\mu} - E^{\mu} u^{\nu} + \epsilon^{\mu\nu\alpha\beta} B_{\alpha} u_{\beta}$$

- We now have the 4 Maxwell equations:

$$\begin{aligned}
\partial_{\mu} E^{\mu} &= \frac{1}{c} J^{\alpha} u_{\alpha} \\
\partial_{\mu} B^{\mu} &= 0 \\
u^{\alpha} \partial_{\alpha} B^{\mu} - \epsilon^{\mu\nu\alpha\beta} u_{\nu} \partial_{\alpha} E_{\beta} &= 0 \\
u^{\alpha} \partial_{\alpha} E^{\mu} + \epsilon^{\mu\nu\alpha\beta} u_{\nu} \partial_{\alpha} B_{\beta} &= -\frac{1}{c} \left(J^{\mu} - u^{\mu} u_{\nu} J^{\nu} \right)
\end{aligned}$$

- These hold in any coordinate system and in any frame.

EM in General Relativity

- We can easily set up EM in a curved, vacuum spacetime.
- Maxwell's equations in terms of the Faraday tensor:

$$\nabla_\mu F^{\mu\nu} = \frac{1}{c} J^\nu$$
$$\nabla_\mu [\tilde{F}^{\mu\nu}] = \nabla_\mu \left[\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \right] = 0$$

- We can perform a similar decomposition as before:

$$E^\mu = F^{\mu\nu} u_\nu$$
$$B^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$$
$$F^{\mu\nu} = E^\nu u^\mu - E^\mu u^\nu + \epsilon^{\mu\nu\alpha\beta} B_\alpha u_\beta$$

- The final system is below:

$$
\begin{aligned}
\bar{D}_\mu E^\mu &= \sigma - 2\omega_\mu B^\mu \\
\bar{D}_\mu B^\mu &= 2\omega_\mu E^\mu \\
\dot{E}_{<\mu>} &= \left(\sigma_{\mu\nu} + \epsilon_{\mu\nu\gamma}\omega^\gamma - \frac{2}{3}\Theta h_{\mu\nu}\right) E^\nu \\
&\quad + \epsilon_{\mu\nu\gamma}\dot{u}^\nu B^\gamma + \mathrm{curl} B_\mu - j_\mu \\
\dot{B}_{<\mu>} &= \left(\sigma_{\mu\nu} + \epsilon_{\mu\nu\gamma}\omega^\gamma - \frac{2}{3}\Theta h_{\mu\nu}\right) B^\nu \\
&\quad - \epsilon_{\mu\nu\gamma}\dot{u}^\nu E^\gamma - \mathrm{curl} B_\mu
\end{aligned}
$$

- Here we have defined:

$$
\begin{aligned}
\bar{D}_\mu X^\nu &\equiv h_\mu^\alpha h_\beta^\nu \nabla_\alpha X^\beta \\
\mathrm{curl} X_\mu &= \epsilon_{\mu\alpha\beta} \bar{D}^\alpha X^\beta \\
\dot{X}_{<\mu>} &= h_\mu^\nu \dot{X}_\nu \\
\dot{X}^\mu &= u^\alpha \nabla_\alpha X^\mu
\end{aligned}
$$

- The fluid kinematic properties arise from the irreducible tensor decomposition of the 4-velocity's covariant derivative:

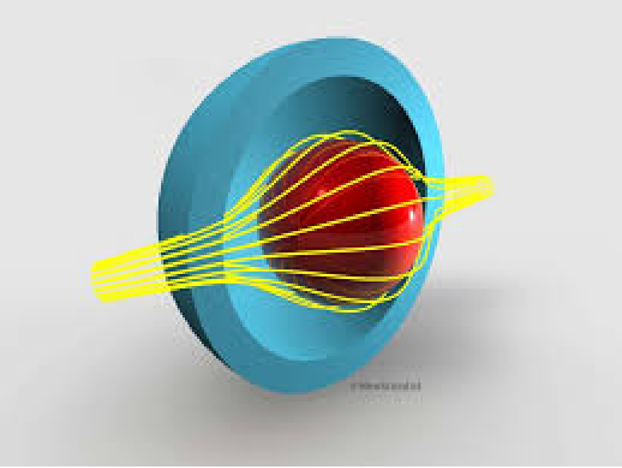
$$\begin{aligned}
\Theta &= \bar{D}_\mu u^\mu \\
\sigma_{\mu\nu} &= \bar{D}_{<\mu} u_{\nu>} \\
\omega_{\mu\nu} &= \bar{D}_{[\mu} u_{\nu]} \\
\omega_\mu &= -\frac{1}{2}\mathrm{curl} u_\mu
\end{aligned}$$

- Physically these represent the
  1. expansion, Θ
  2. shear tensor, $\sigma_{\mu\nu}$
  3. vorticity tensor, $\omega_{\mu\nu}$
  4. vorticity vector, ω_μ

Transformation Optics

- Tailor-made meta-materials with unusual optical properties are possible
- Materials with negative refractive indices; cloaking devices
- Study initiated by Pendry and Leonhardt independently in 2006.
- Exploits older results by Tamm (1924) and Plebanski (1960)

Cloaking devices



- Key insights follow from these identities:

$$
\begin{aligned}
\nabla_\alpha X^\alpha &\equiv \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} X^\alpha \right) \\
M^{\alpha\beta} &= -M^{\beta\alpha} \\
\Rightarrow \nabla_\alpha M^{\mu\alpha} &\equiv \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} M^{\mu\alpha} \right)
\end{aligned}
$$

- EM in curved, vacuum spacetimes is equivalent to EM in flat, dielectric spacetimes!
- Curvature manifests as an effective permittivity and permeability.

- Decompose the covariant Maxwell equations.

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= \frac{1}{c} J^\nu \\ \Rightarrow \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} F^{\mu\nu} \right) &= \frac{1}{c} J^\nu \\ \Rightarrow \partial_\mu \left(\sqrt{-g} F^{\mu\nu} \right) &= \frac{1}{c} \left(\sqrt{-g} J^\nu \right) \end{aligned}$$

- We now re-scale the field strength tensor and 4-current density:

$$\begin{aligned} M^{\mu\nu} &\equiv \sqrt{-g} F^{\mu\nu} \\ j^\nu &\equiv \sqrt{-g} J^\nu \end{aligned}$$

to obtain

$$\partial_\mu M^{\mu\nu} = \frac{1}{c} j^\nu$$

which resembles the special relativity form.

- The curved spacetime, vacuum Maxwell equations is equivalent to the flat spacetime, dielectric system:

$$\begin{aligned}\partial_\mu D^\mu &= \rho\\ \partial_\mu B^\mu &= 0\\ u^\alpha \partial_\alpha B^\mu &= -\epsilon^{\mu\alpha\beta}\partial_\alpha E_\beta\\ u^\alpha \partial_\alpha D^\mu &= \epsilon^{\mu\alpha\beta}\partial_\alpha E_\beta - j^\mu\end{aligned}$$

- We impose the constitutive relations:

$$\begin{aligned}D^\beta &= \epsilon_0 \epsilon^{\beta\alpha} E_\alpha + \frac{1}{c}\epsilon^{\beta\gamma\lambda} w_\gamma H_\lambda\\ B^\beta &= \mu_0 \mu^{\beta\alpha} H_\alpha - \frac{1}{c}\epsilon^{\beta\gamma\lambda} w_\gamma E_\lambda\end{aligned}$$

with $w_i = g_{0i}/g_{00}$.

- The effective spacetime permittivity and permeability are

$$\epsilon^{\alpha\beta} = \mu^{\alpha\beta} = -\frac{\sqrt{-g}}{g_{00}} g^{\alpha\beta}$$

- The constitutive relations can be written in index-free notation:

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E} + \frac{\mathbf{w}}{c} \times \mathbf{H}$$
$$\mathbf{B} = \mu_0 \mu \mathbf{H} - \frac{\mathbf{w}}{c} \times \mathbf{E}$$

- Note that here ϵ and μ are tensors defined via

$$\epsilon^{ij} = \mu^{ij} = \frac{\sqrt{-g}}{g_{00}} g^{ij}$$
$$w_i = \frac{g_{0i}}{g_{00}}$$

Comments

- The electric permittivity and magnetic permeability are identical; curved spacetime behaves as an impedance-matched medium.
- The permittivity and permeability are matrices; curved spacetime behaves as an anisotropic medium.
- We should be able to obtain an equivalent formulation for geometric and physical optics in curved spacetimes.
- Revisit electromagnetic phenomena in astrophysics eg. the Blandford-Znajek mechanism, superradiance.
- The formalism is geometric and applicable for any given metric.
- Electromagnetic waves carry imprints of gravitational wave backgrounds.
- Gertshenstein effect: EM-GW mixing in the presence of a magnetic field in curved spacetime

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