Quasi-normal and Quasi-resonant modes in Metric $f(R)$ gravity

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Motivations

- After 2016, there has been a great deal of interest in modified gravity
- The interest has not had much tangible impact
- Lack of realiable exact, linearized results to help test new codes

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General Relativity and Quantum Cosmology

[Submitted on 2 Aug 2020 (v1), last revised 29 Nov 2020 (this version, v2)]

Strongest constraint in $f(R) = R + \alpha R^2$ gravity: stellar stability

Juan M. Z. Pretel, Sergio E. Jorás, Ribamar R. R. Reis

In the metric approach of $f(R)$ theories of gravity, the fourth-order field equations are often recast as effective Einstein equations in the presence of standard matter and a curvature fluid (which gathers all the extra terms), always in the Jordan frame. In this picture, we investigate the strong gravity regime of the $f(R) = R + \alpha R^2$ model. In particular, we focus on the stability of a compact star composed by a mixture of ordinary matter -- described by a polytropic equation of state -- and an effective curvature fluid in an otherwise standard Einstein gravity, so that we are able to apply the usual equations that govern the radial adiabatic oscillations of relativistic stars. Our new restriction on the free parameter is $\alpha \lesssim 2.4 \times 10^8\ \mathrm{cm}^2$ in order to guarantee stellar stability, about 100 times more restrictive than previous results (based on mass-radius relations alone) in the literature.

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General Relativity and Quantum Cosmology

[Submitted on 6 Aug 2009]

Probing the f(R) formalism through gravitational wave polarizations

Marcio E.S. Alves, Oswaldo D. Miranda, Jose C.N. de Araujo

The direct observation of gravitational waves (GWs) in the near future, and the corresponding determination of the number of independent polarizations, is a powerful tool to test general relativity and alternative theories of gravity. In the present work we use the Newman-Penrose formalism to characterize GWs in quadratic gravity and in a particular class of f(R) Lagrangians. We find that both quadratic gravity and the f(R) theory belong to the most general invariant class of GWs, i.e., they can present up to six independent polarizations of GWs. For a particular combination of the parameters, we find that quadratic gravity can present up to five polarizations states. On the other hand, if we use the Palatini approach for f(R) theories, GWs present only the usual two transverse-traceless polarizations such as in general relativity. Thus, we conclude that the observation of GWs can strongly constrain the suitable formalism for these theories.

Metric $f(R)$

Action:

$$S = \frac{1}{2\kappa^2}\int dx^4\left[\sqrt{-g}f(R) + 2\kappa^2\mathcal{L}_m\right]$$

Field Equations

$$\Sigma_{ab} = \kappa^2 T_{ab}$$

$$\begin{aligned}\Sigma_{ab} &= f'R_{ab} - \tfrac{1}{2}fg_{ab} - \nabla_a\nabla_b f' + g_{ab}\Box f' \\ &= f'R_{ab} - \tfrac{1}{2}fg_{ab} - f''\nabla_a\nabla_b R - f'''\nabla_b R\nabla_a R + g_{ab}(f'''\nabla^c R\nabla_c R + f''\Box R)\end{aligned}$$

$$3\Box f' - 2f + f'R = \kappa^2 T$$

Bianchi Identities

$$\nabla^a\Sigma_{ab} = 0 = \nabla^a T_{ab}$$

The QNM problem

Given some eignevalue problem described by a master equation, we have

- Quasi-normal modes
- The algebraically special mode (tensors only)
- Quasi-bound modes (scalars only)

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  - In the limit $\Im\omega \to 0^-$, we talk of Quasi-resonant modes

Boundary conditions

- Horizon: Ingoing
- Infinity: Decaying

Initial value problem

System 1

$$u_{tt} = c^2 u_{xx}$$

System 2

$$\partial_t u = v$$
$$\partial_t v = \partial_{xx} u$$

System 3

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

System 4

$$u_{\xi\eta} = 0$$
$$\xi = x + ct,\, \eta = x - ct$$

System 5

$$u_t = s$$
$$s_t = cw_x$$
$$w_t = cs_x$$

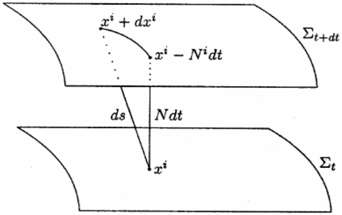
Initial value problem

- 3+1 Cauchy

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i\mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \ ,$$

- Characteristic

$$ds^2 = -\left(e^{2\beta}\left(1+\frac{W}{r}\right) - r^2 h_{AB}U^A U^B\right)\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r - 2r^2 h_{AB}U^B\mathrm{d}u\mathrm{d}x^A + r^2 h_{AB}\mathrm{d}x^A\mathrm{d}x^B \ .$$



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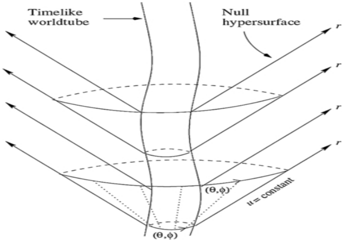

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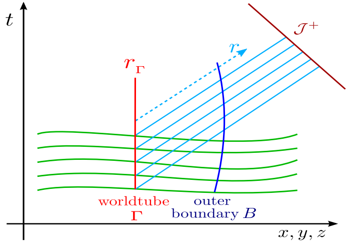

Bondi-Sachs system

The field equations result in a hierarchy of equations

- Hypersurface equations (4 equations)
- Evolution equations (2 equations)
- Trivial equation (1 equation)
- Supplementary equations (3 equations)

Introduce a complex dyad q^A, with $q^A q_A = 0$, $q^A \bar{q}_A = 2$, then define

$$J = \tfrac{1}{2} q^A q^B h_{AB} \qquad \text{and} \qquad U = U^A q_A \qquad \text{etc}$$

Linearized perturbations

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\mathsf{d}u^2 - 2\mathsf{d}u\mathsf{d}r + r^2 q_{AB}\mathsf{d}x^A\mathsf{d}x^B \ ,$$

$$R, J, \bar{J}, U, \bar{U}, w, \beta = \mathcal{O}(\epsilon) \ ,$$

$$f(R) = f'_{(0)}\, R \ ,$$

Harmonic decomposition

In general

$$f(u,r,\Omega) = \sum_{\ell m} f_{\ell m}(u,r)\,{}_sZ_{\ell m}\;.$$

For a spherically symmetric background, the (ℓ, m) modes decouple

$$f(u,r,\Omega) = f(u,r)\,{}_sZ_{\ell m}$$

One can further decompose $f(u,r)$

$$f(u,r,\Omega) = \hat{f}(r)\Re(e^{\rho u})\,{}_sZ_{\ell m}$$

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For a spherically symmetric background, the (ℓ, m) modes decouple

$$f(u,r,\Omega) = f(u,r)\,{}_sZ_{\ell m} \qquad \text{time domain}$$

One can further decompose $f(u,r)$

$$f(u,r,\Omega) = \hat{f}(r)\Re(e^{\rho u})\,{}_sZ_{\ell m} \qquad \text{frequency domain}$$

$$\partial_u f \longleftrightarrow \rho\,\hat{f}(r)$$

Master equations

- Tensor sector, define $\mathcal{J} = r^3(rJ(r))_{,rr}$
  Frequency domain

$$r^2\,(r-2M)\,\mathcal{J}_{,rr} - 2r\left(\rho r^2 + r - 5M\right)\mathcal{J}_{,r} - \left[2\rho r^2 + (\ell^2+\ell-2)r + 16M\right]\mathcal{J} = 0\,.$$

- Scalar sector
  Frequency domain

$$r\,(r-2M)\,R_{rr} - 2\left(\rho r^2 - r + M\right)R_r - \left[2\rho r + \ell(\ell+1) + m^2 r^2\right]R = 0\,.$$

Master equations

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  $$r^2\,(r-2M)\,\mathcal{J}_{,rr} - 2r^3\mathcal{J}_{ur} - 2r\,(r-5M)\,\mathcal{J}_{,r} - 2r^2\mathcal{J}_u - \left[(\ell^2+\ell-2)r + 16M\right]\mathcal{J} = 0\,.$$

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Continued fraction method: Tensor modes

- *Near the horizon* $r = 2M$. The two independent local behaviours are

$$\mathcal{J} \sim (r-2M)^0, \qquad \text{and} \qquad \mathcal{J} \sim (r-2M)^{4\rho M-2}. \tag{1}$$

- *Asymptotically as* $r \to \infty$. The two independent behaviours are

$$\mathcal{J} \sim \frac{1}{r}, \qquad \text{and} \qquad \mathcal{J} \sim r^{3+4\rho M}\, e^{2\rho r}. \tag{2}$$

The QNM ansatz

$$\mathcal{J} = r^{4\rho M+3}\, e^{2\rho(r-2M)} \sum_{n=0}^{\infty} a_n \left(\frac{r-2M}{r}\right)^n . \tag{3}$$

Continued fraction method: Tensor modes

$$
\begin{aligned}
\alpha_0 a_1 + \beta_0 a_0 &= 0\,, \\
\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} &= 0 \qquad n = 1, 2, \cdots
\end{aligned}
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with

$$\begin{aligned}\alpha_n &= -n^2 + (4M\rho - 4)\, n + 4M\rho - 3 \,, \\ \beta_n &= 2n^2 - (16M\rho - 2)\, n + 32M^2\rho^2 - 8M\rho - 3 + \ell(\ell+1) \,, \\ \gamma_n &= -n^2 + (8M\rho + 2)\, n - 16M^2\rho^2 - 8M\rho \,,\end{aligned}$$

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This is equivalent to

$$0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \cdots$$

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Which can be inverted any number of times

$$\left[\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} -} \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} -} \cdots \frac{\alpha_0 \gamma_1}{-\beta_0} \right] = \frac{\alpha_n \gamma_{n+1}}{\beta_{n+1} -} \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} -} \frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3} -} \cdots$$

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Remainder term

$$R_N \sim C_0 + \frac{C_1}{\sqrt{N}} + \frac{C_2}{N} + \mathcal{O}\left(N^{-3/2}\right),$$

Continued fraction method: Scalar modes

- *Near the horizon* $r = 2M$. The two independent local behaviours are

$$R \sim (r - 2M)^0, \qquad \text{and} \qquad R \sim (r - 2M)^{4\rho M}. \tag{4}$$

- *Asymptotically as* $r \to \infty$. The two independent behaviours are

$$R_\infty^{(\pm)}(r) \ \sim \ \exp\big[(\rho \pm \kappa)\, r\big]\ r^{\sigma_\pm} \tag{5}$$

where

$$\kappa = \sqrt{\rho^2 + m^2} \qquad \text{and} \qquad \lambda_\pm = \rho \pm \kappa \qquad \text{and} \qquad \sigma_\pm = -1 \ \pm \ \frac{M}{\kappa}\,\lambda_\pm^2 \tag{6}$$

The QNM ansatz

$$R = e^{(\rho+\kappa) r}\, r^{\sigma_+} \sum_{n=0}^{\infty} a_n \left(1 - \frac{2M}{r}\right)^n, \tag{7}$$

Continued fraction method: Scalar modes

$$
\begin{aligned}
\alpha_0 a_1 + \beta_0 a_0 &= 0\,, \\
\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} &= 0 \qquad n = 1, 2, \cdots
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Continued fraction method: Scalar modes

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\end{aligned}
$$

with

$$
\begin{aligned}
\alpha_n &= (n+1)\big(n+1-4M\rho\big) \,, \\
\beta_n &= -2n^2 + \big(4M\lambda_+ + 2\sigma_+\big)\, n + \big(\sigma_+ - 4M\sigma_+\lambda_+ - 2M\lambda_+ - \ell(\ell+1)\big) \,, \\
\gamma_n &= \big(n-1-\sigma_+\big)^2 \,.
\end{aligned}
$$

Continued fraction method: Scalar modes

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This can again be written in terms of a continued fraction

$$\left[\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1}-}\,\frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2}-}\cdots\frac{\alpha_0\gamma_1}{-\beta_0}\right] = \frac{\alpha_n\gamma_{n+1}}{\beta_{n+1}-}\,\frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-}\,\frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-}\cdots$$

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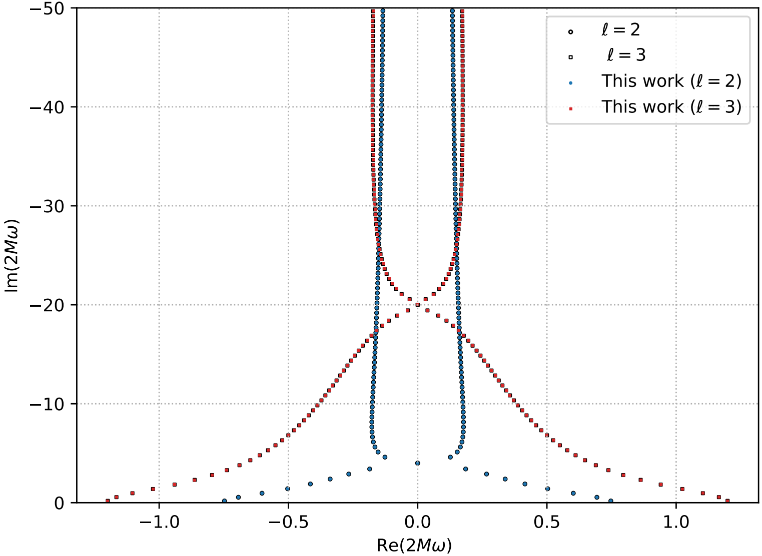
Remainder term

$$R_N \sim C_0 + \frac{C_1}{\sqrt{N}} + \frac{C_2}{N} + \mathcal{O}\left(N^{-3/2}\right),$$

Results: Tensor modes



Time domain: Tensor and Scalar modes

$$r^2\,(r-2M)\,\mathcal{J}_{,rr} - 2r^3\mathcal{J}_{ur} - 2r\,(r-5M)\,\mathcal{J}_{,r} - 2r^2\mathcal{J}_u - \left[(\ell^2+\ell-2)r + 16M\right]\mathcal{J} = 0\;. \tag{8}$$

Introduce $\phi = \mathcal{J}_u$,

$$\mathcal{J}_u = \phi \tag{9}$$

$$\phi_r + \frac{1}{r}\phi = \frac{r-2M}{2r}\,\mathcal{J}_{,rr} - \frac{r-5M}{r^2}\,\mathcal{J}_{,r} - \frac{(\ell^2+\ell-2)\,r+16M}{2r^3}\,\mathcal{J}. \tag{10}$$

Time domain: Tensor and Scalar modes

$$r\,(r-2M)\,R_{rr} - 2r^2 R_{,ur} - 2\,(-r+M)\,R_r - 2rR_u - \left[\ell(\ell+1) + m^2 r^2\right] R = 0\,. \tag{8}$$

Define $\psi = R_u$

$$\begin{aligned} R_u &= \psi \quad (9) \\ \psi_r + \frac{1}{r}\psi &= \frac{r-2M}{2r} R_{,rr} + \frac{r-M}{r^2} R_{,r} - \frac{\ell(\ell+1)+m^2r^2}{2r^2} R. \quad (10) \end{aligned}$$

Time domain: Tensor and Scalar modes

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Both equations take the form

$$P_u = Q \tag{11}$$

$$Q_r = -\frac{1}{r}Q + S(r,P,P_r,P_{rr}) \tag{12}$$

Ongoing

- Are there other ways of deriving the algebraically special mode? (e.g. can one find explicit conditions under which the operator on $\mathscr{I}$ is self-adjoint?)
- Long term evolutions of the scalar equations.