Numerical modeling of GW sources

Denis Pollney

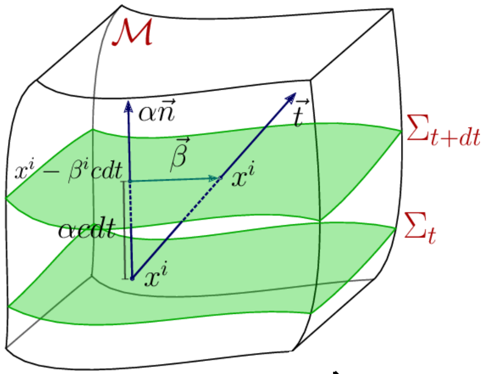
Department of Mathematics, Rhodes University

Overview

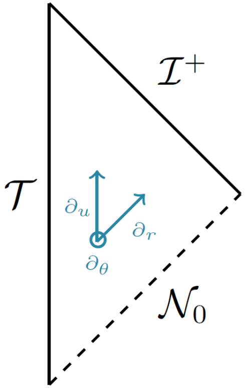
- Improved formulations for null evolutions
- New observables at scri
- What goes on inside BHs

Formulations for null evolutions

Numerical approaches to modelling spacetime



- **3+1 (ADM) methods**:
  - **Flexible**: No symmetry required
  - **Familiar**: Methodologies similar to computational fluid dynamics
  - **Expensive**: High grid resolution + large grids
  - **Potentially unstable**: Short evolution before crash
- **Characteristic methods**:
  - Based on **Bondi coordinates**
  - **Stable**: Demonstrated for single BHs by Winicour et al. (1997)
  - **Asymptotic**: Compactified null coordinates allow for calculations at null infinity
  - **Caustics**: Coordinates are based on null surfaces which tend to fold on themselves
  - **Axisymmetric**: Coordinates follow outgoing rays have difficult with multiple sources



3+1 (ADM) approach - Relativity as an evolution problem

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

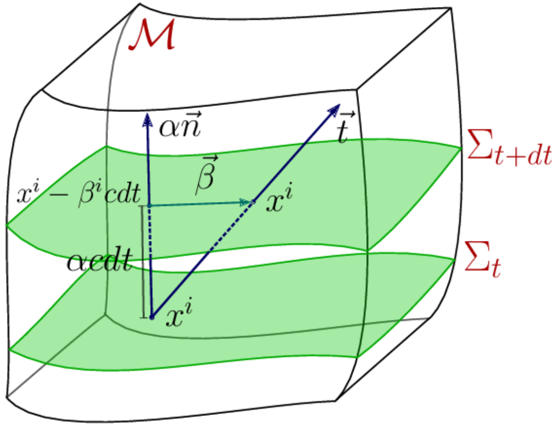
- **Spacelike slices**: Define a foliation Σ_t with normal n^μ.
- **Field equations**: Project the Einstein equations onto Σ_t to give evolution equations:

$$\mathcal{L}_n \gamma_{\mu\nu} = 2K_{\mu\nu},$$
$$\mathcal{L}_n K_{\mu\nu} = -D_\mu D_\nu \alpha + \alpha\big(R_{\mu\nu} + KK_{\mu\nu} - 2K_{\mu\alpha}K^\alpha{}_\nu\big).$$

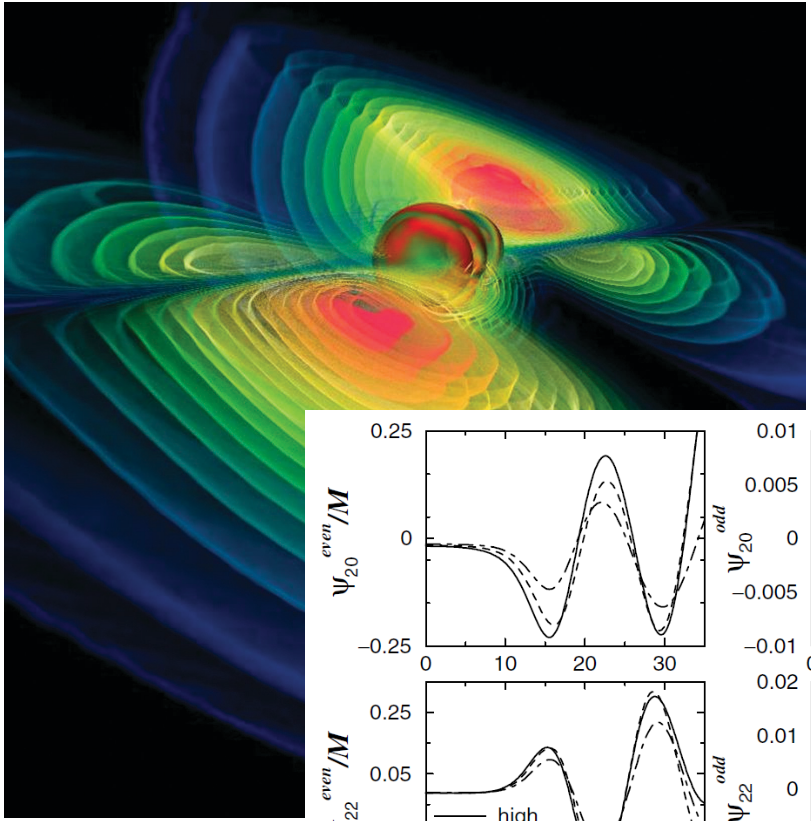
in terms of variables

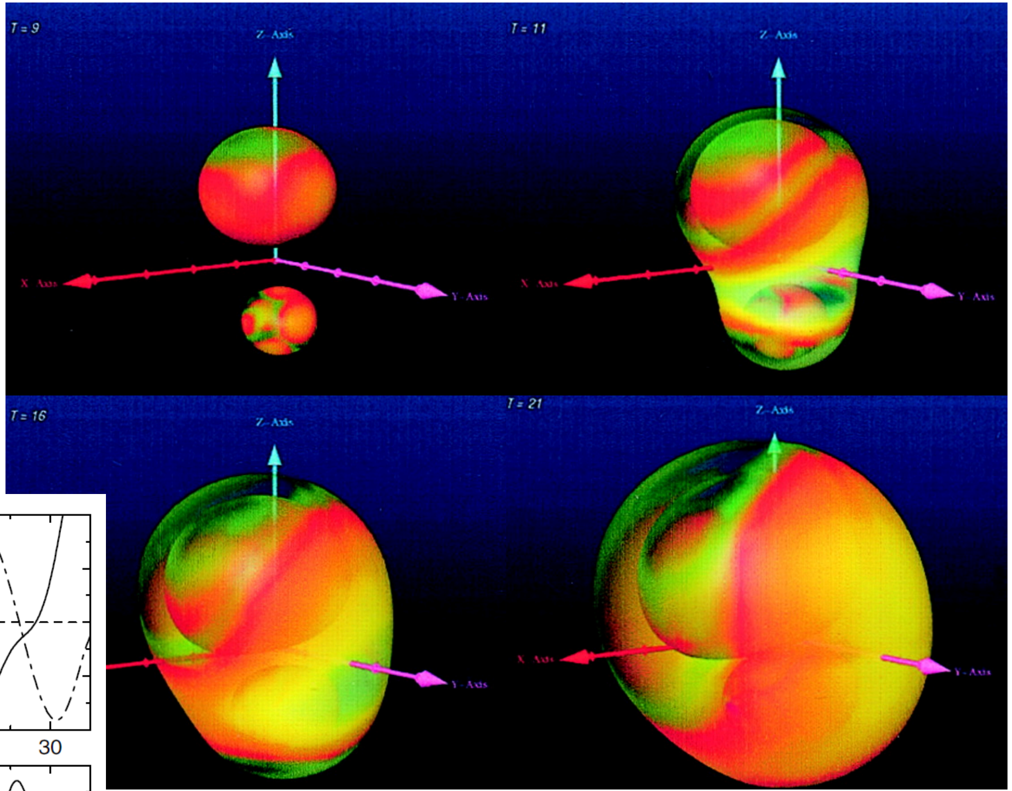
- $\gamma_{\mu\nu}$ – the 3-metric on Σ_t
- $K_{\mu\nu}$ – the extrinsic curvature of Σ_t
- α, β^i - lapse and shift defining the choice of normal, n^μ

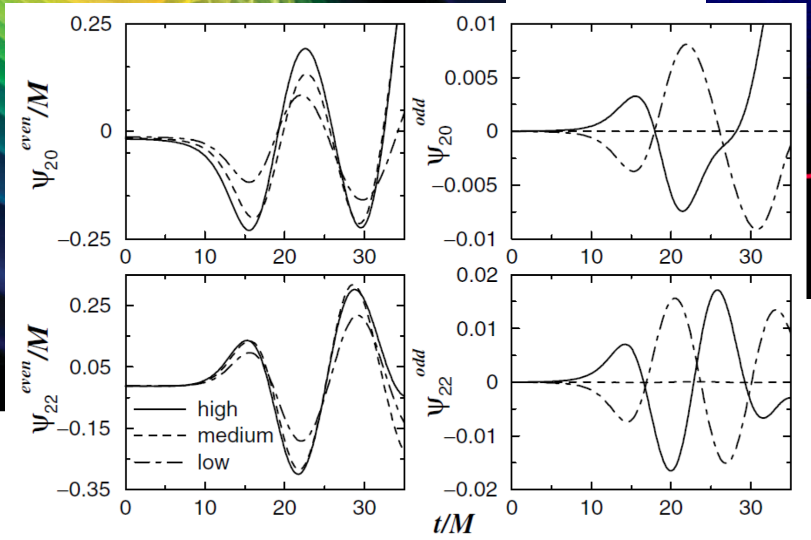
(There are also 4 constraint equations.)



Early results - Frustrating instabilities







Alcubierre et al. (2001) “Grazing Collision”

Hyperbolic formulations ⇒ Longer evolutions

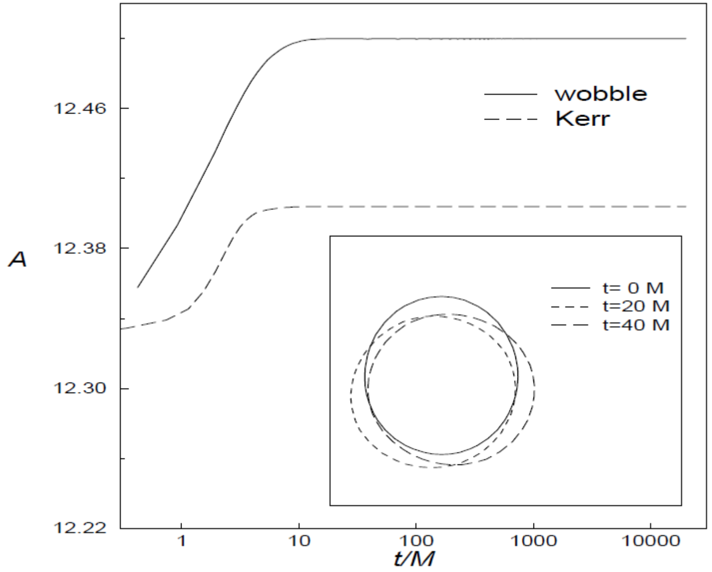
- **Slow progress**: Researchers began to notice better stability using alternate formulations of ADM:
  - **1995 Oohara, Kojima, Nakamura**: Introduced auxiliary variables $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}$.
  - **1997 Bona, Masso**: Developed a hyperbolic variant of ADM
  - **1998 Baumgarte, Shapiro**: Showed improved stability of BH evolutions using the OKN variables
  - **2001 Alcubierre, Bruegmann**: First long-term stable evolution of a single BH using "BSSN" variables
- **Pattern**: The stability of evolutions depends strongly on the choice of variables.
  - **Sarbach, Tiglio, et al. (2005)**: The ADM system is weakly hyperbolic.

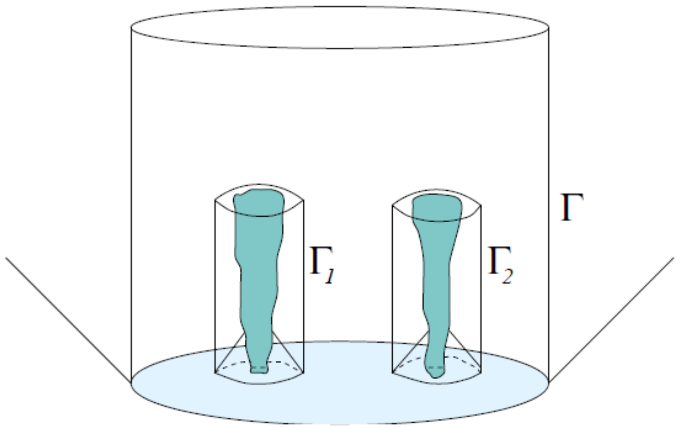
$$\text{Strongly hyperbolic PDEs} \quad \Leftrightarrow \quad \text{Well-posed IBVP} \quad \Leftrightarrow \quad \text{Numerical stability}$$

- **2007 Pretorius**: Long-term stable evolution of a BBH using harmonic coordinates.

Characteristic methods

- Pioneered by Winicour in 1990s.
- Based on a Bondi-Sachs coordinate system.
- Leads to a heirarchy of evolution equations solved in turn.
- Impractical for use in a binary BH problem:
  - Coords based on outgoing null rays → spherical topology
  - Caustics form in strong-field regions





Bondi evolution system

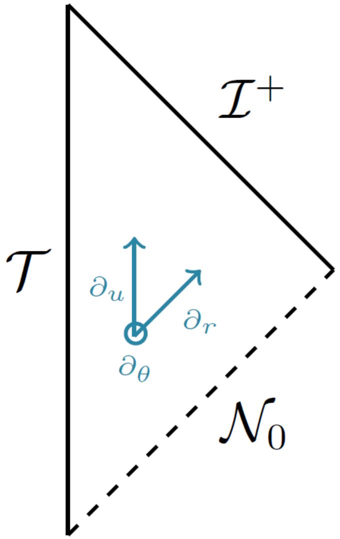
- Coordinates: (u, r, θ, φ) – u is null:

$$\mathrm{d}s^2 = -\left(\frac{V}{r}e^{2\beta} + U^2 r^2 e^{2\gamma}\right)\mathrm{d}u^2 - 2e^{2\beta}\,\mathrm{d}u\,\mathrm{d}r + 2Ur^2e^{2\gamma}\,\mathrm{d}u\,\mathrm{d}\theta + r^2\left(e^{2\gamma}\,\mathrm{d}\theta^2 + e^{-2\gamma}\sin^2\theta\,\mathrm{d}\varphi^2\right).$$

- Metric functions in spherical symmetry: (β, U, V, γ)

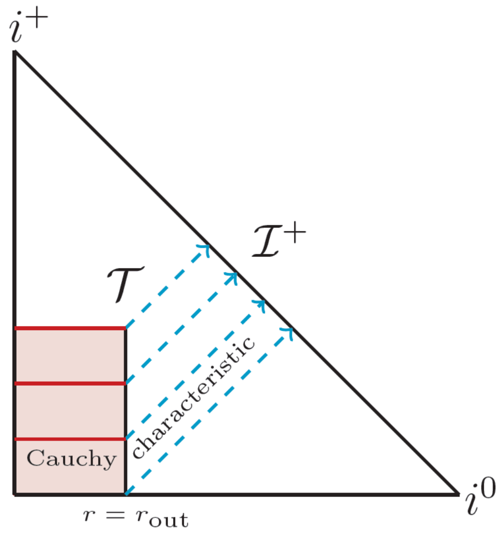
- Main Einstein equations:

$$\begin{aligned}
\partial_r\beta &= F_1(\gamma), \\
\partial_r^2 U &= F_2(\gamma, \beta), \\
\partial_r V &= F_3(\gamma, \beta, U), \\
\partial_u\gamma &= F_4(\gamma, \beta, U, V).
\end{aligned}$$

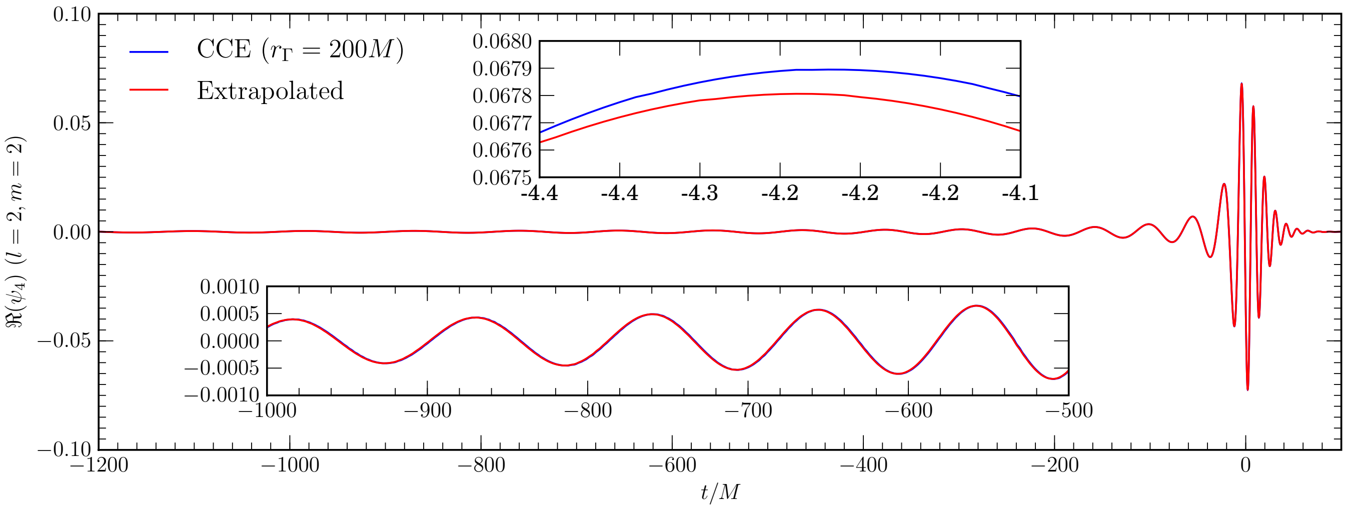


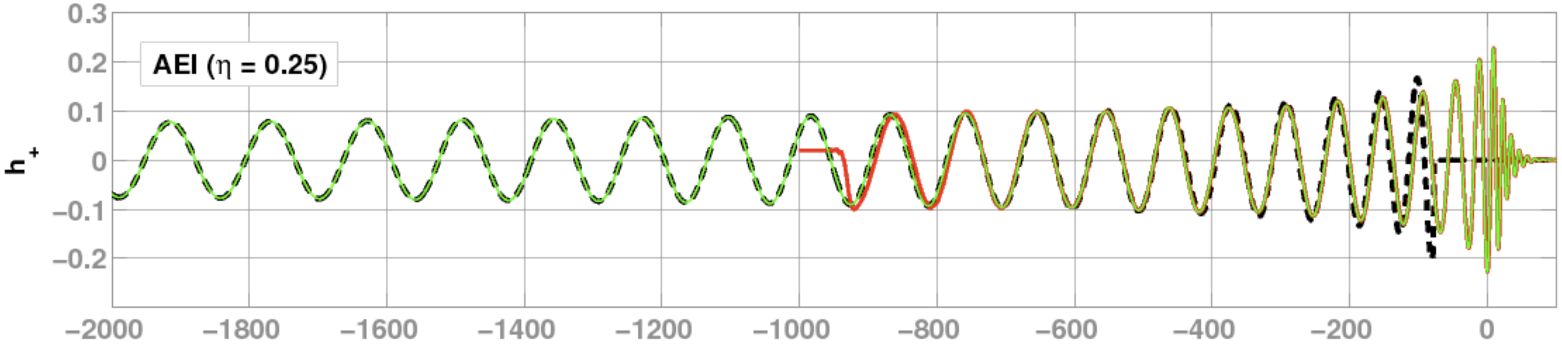
Best of both worlds - Characteristic extraction and matching

- 3+1 evolutions are ideal for the near-zone of an isolated source.
- Characteristic evolutions are ideal for modelling wave-like behaviour and can include $\mathcal{J}^+$
- **Characteristic extraction (CCE)**:
  - A binary system is modelled using a 3+1 evolution.
  - Data is transcribed onto a timelike world-tube at a fixed radius from the source
  - The data is transported to $\mathcal{J}^+$ using a characteristic evolution
- **Characteristic matching (CCM)**:
  - Data is transferred in both directions across the world-tube
  - The characteristic code provides a “boundary condition” for the 3+1 code.



Gravitational waves from CCE





Problems with CCM

- Characteristic matching transfers data between Cauchy (3+1) and characteristic evolutions:
  - The 3+1 evolution provides inner boundary data for the characteristic evolution.
  - They characteristic evolution provides outer boundary data for the 3+1 evolution.
- Attempts at implementation have proven unsuccessful with instabilities reminiscent of early attempts at 3+1 via ADM variables
- Recent work has uncovered a suprising result:
  - Giannakopoulos, Hilditch, et al. (2022-23):
    - The PDE system is weakly hyperbolic in the Bondi gauge.
    - The characteristic IBVP is ill-posed in the L^2 norm.
    - The system may be weakly well-posed in another norm (open question)

- Can null evolutions be re-formulated as a strongly hyperbolic PDE problem?
- A powerful approach to developing hyperbolic systems with spacelike slicings are the $Z4$ formulations (Bona, Palenzuela, Ledvinka 2003-4).
- The Einstein equations are expanded by introducing a new 4-vector, Z_μ:

$$R_{\mu\nu} + \nabla_{(\mu} Z_{\nu)} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

- **Constraint recovery**:
  - $Z_\mu = 0$ recovers the standard Einstein equations
  - The ADM constraints become evolution equations for the Z_μ.
  - The degree to which the Z_μ deviates from zero becomes a measure of numerical error.

Harmonic coordinates

- The Einstein equations can be expanded in terms of the metric $g_{\mu\nu}$ as

$$-\Box g_{\mu\nu} + \partial_{(\mu}\Gamma_{\nu)} = \dots$$

where $\Gamma^{\nu} = g^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta}$.

- A coordinate system x^{μ} is harmonic if each coordinate function satisfies the wave equation

$$\Box x^{\mu} = 0 \qquad \Longleftrightarrow \qquad \Gamma^{\mu} = g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0.$$

- For these coordinates, the Einstein equations reduce to a wave equation for $g_{\mu\nu}$:

$$-\Box g_{\mu\nu} = \dots$$

- In harmonic coordinates the wave equation is manifestly strongly hyperbolic.

Harmonic coordinates and $Z4$

- The $Z4$ equations expand as

$$-\Box g_{\mu\nu} + \partial_\mu(\Gamma_\nu + 2Z_\nu) + \partial_\mu(\Gamma_\nu + 2Z_\nu) = \ldots$$

- We find that the system reduces to a wave equation when

$$-\Gamma_\mu = 2Z_\mu,$$

  and harmonic coordinates are recovered when $Z_\mu = 0$.

- In this sense the Z_μ act as "gauge source" functions for the system, controlling the evolution of the coordinates.

- By placing various restrictions on the Z_μ, Bona et al. have shown that various popular 3+1 evolution schemes (ADM, BSSN, KST, harmonic) can be seen as classes of $Z4$ systems

Harmonic coordinates vs Bondi coordinates

- The gauge source functions for the Bondi metric variables are:

$$
\begin{aligned}
\Gamma^0 &= -\frac{2e^{-2\beta}}{r}, \\
\Gamma^1 &= \frac{e^{-2\beta}\left(r(\partial_r V - r\partial_\theta U) - r^2 \cot\theta U + V\right)}{r^2}, \\
\Gamma^2 &= \frac{2e^{-2\gamma}(\partial_\theta \beta - \partial_\theta \gamma + \cot(\theta)) - e^{-2\beta}\left(r^2 \partial_r U + 2rU\right)}{r^2}.
\end{aligned}
$$

- It is clear that the Bondi coordinates are incompatible with harmonic coordinates (except asymptotically). For instance,

$$
\Gamma^0 = -\frac{2e^{-2\beta}}{r} = 0 \qquad \Longrightarrow \qquad \beta \to \infty
$$

- With Jonathan Gouws, we are looking at generalizations of Bondi coordinates that satisfy strong hyperbolicity.

"New" observables at $\mathcal{J}^+$

Newman-Penrose constants

- The NP constants, G_m, are five absolutely conserved quantities defined at $\mathscr{I}^+$.
- They arise from the asymptotic expansion of Ψ_0 (which describes "ingoing" gravitational radiation):

$$\Psi_0 = \frac{\Psi_0^{(0)}}{r^5} + \frac{\Psi_0^{(1)}}{r^6} + \mathcal{O}(r^{-7}).$$

- The constants are defined by integrating over a 2-sphere at infinity:

$$G_m = \oint {}_2Y_{2m} \Psi_0^{(0)} \, \mathrm{d}\Omega$$

where ${}_2Y_{2m}$ are the spin-weight $s = 2$ spherical harmonics.

Invariance and symmtry

- **BMS invariance**: The NP constants are invariant under the Bondi-Metzner-Sachs group, specifically supertranslations.
- **Conservation**: Unlike the Bondi mass, which decreases as GWs carry energy across $\mathcal{J}^+$, the NP constants remain the same regardless of the "cut" (time slice) of $\mathcal{J}^+$ at which they are measured.
- **Independence**: They represent information about the gravitational field that cannot be changed by the emission of radiation.

Physical interpretation

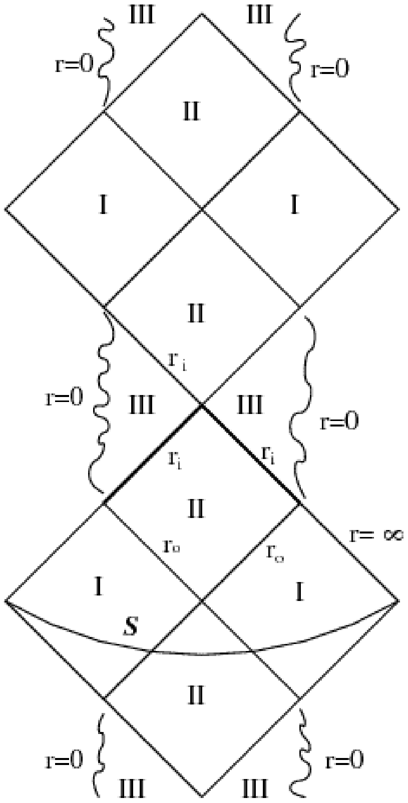
- **Stationary systems**: The NP constants are directly related to the multipole moments of the source (mass, angular momentum)
- **Radiating systems**: In the presence of gravitational waves, they encode the “residual” information about the source after the wave has passed.
- **Gravitational memory effect**: The NP constants can be related to the permanent displacement of test particles after a GW passes, known at the “non-linear memory” or “Christodoulou memory.”

Numerical evaluation

- Recently Camden et al. (2025) performed numerical simulations evaluating NP constants:
  - Used Friedrich's General Conformal Field Equations
  - conformal Gauss gauge
- For a perturbed BH, they confirmed their methodology conserves the G_m.
- Open questions as to how these constants can be interpreted in highly dynamical spacetimes.

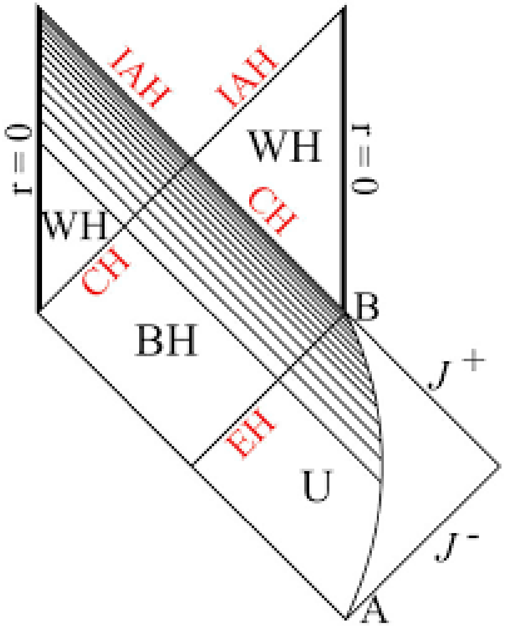
Inside black holes

- **Mass inflation**: A process that occurs in spacetimes with an “inner” Cauchy horizon (e.g. Reissner-Nordstrom, Kerr).
- **Cauchy horizon**: A Cauchy horizon represents the limit of predictability in a spacetime.
- **Instability** Classically, the Cauchy horizon was thought to be a smooth boundary.
  - Poisson, Israel (1990) demonstrated that it is inherently unstable.
  - A strong curvature singularity develops on the horizon

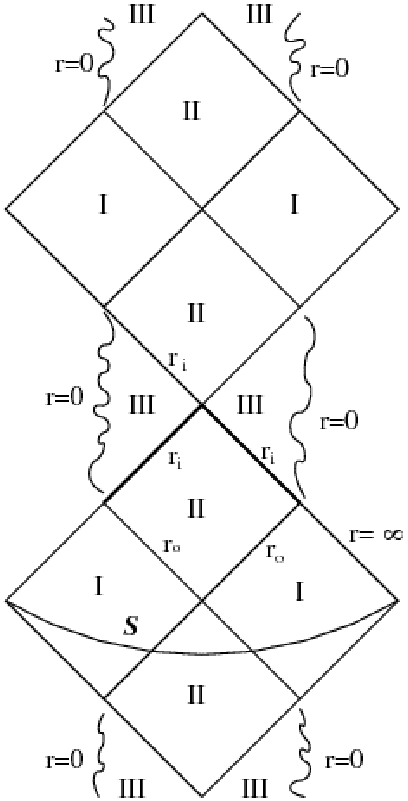


Mechanism

- **Infalling radiation (tail)**: Gravitational waves from the external universe fall into the BH
- **Outgoing radiation (back-scattered)**: A small fraction of the radiation is back scattered by the curvature
- **Collision**: The blue-shifted infalling stream "collides" with the outgoing stream at the horizon.
- **"Inflation"**: The radiation interaction causes the mass function $m(u, v)$ to grow exponentially.



- **Singularity formation**: The exponential growth of the mass function causes a strong curvature singularity to form at the Cauchy horizon
  - Open questions as to the nature of the singularity (spacelike or null)
- **Strong cosmic censorship**: This phenomenon supports "strong cosmic censorship"
  - "Nature abhors a Cauchy horizon" and prevents the formation of regions where predictability breaks down



Dynamical black hole interiors

- **Analytical work** has been based largely on the Reissner-Nordstrom spacetime.
  - Calculations much more complicated in Kerr
- **Numerical work** There is sparse numerical work on black hole interiors and numerous technical challenges:
  - Approach to a strong curvature singularity
  - Coordinate difficulties - e.g. caustics
- **Preliminary results**:
  - Stephanie Dwyer (Rhodes) has developed the **ingoing Teukolsky equation** in the **Doran (2001)** form of the Kerr metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(\frac{\rho}{\sqrt{r^2+a^2}}\,\mathrm{d}r + \alpha(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)\right)^2 - \rho^2\,\mathrm{d}\theta^2 - (r^2+a^2)\sin^2\theta\,\mathrm{d}\varphi^2.$$

  - These are **horizon-penetrating coordinates** (unlike Boyer-Lindquist) allowing the perturbations to be evaluated between the horizons