RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 1 : SEPTEMBER 2011

MATHEMATICS HONOURS

GEOMETRIC CONTROL THEORY

AVAILABLE MARKS : 52
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**24** marks)

(a) Define the terms *Minkowski plane, Lorentz transformation* and *Lorentzian matrix.* Hence prove that, given a 2×2 matrix A, the following statements are equivalent.

i. A is Lorentzian.

ii. $A^\top \mathbb{J} A = \mathbb{J}$, where $\mathbb{J} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

iii. The columns of A form a Lorentz orthonormal basis for $\mathbb{R}^{1,1}$.

iv. The rows of A form a Lorentz orthonormal basis for $\mathbb{R}^{1,1}$.

v. $A \mathbb{J} A^\top = \mathbb{J}$, where $\mathbb{J} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Define the group (of matrices) $\mathsf{O}(1,1)$ and then show that $\mathsf{O}(1,1)$ is a *matrix Lie group.* Is the group $\mathsf{O}(1,1)$ *connected* ? Make a clear statement and then prove it.

(c) Determine the *connected component* of the identity in $\mathsf{O}(1,1)$. Find the *Lie algebra* of this matrix Lie group.

[10,8,6]

Question 2. (**28** marks)

Let $\Sigma = (\mathsf{G}, \Gamma)$ be a left-invariant control system. (It is assumed that the state space G is a *matrix Lie group* and that the admissible controls are *piecewise-constant maps* $u(\cdot) : [0,T] \to U \subseteq \mathbb{R}^\ell$.)

(a) Define the term *trajectory* and then show that for any trajectory $g(\cdot) : [0,T] \to \mathsf{G}$ with $g(0) = g_0$, there exist $N \in \mathbb{N}$ and

$$\tau_1, \tau_2, \ldots, \tau_N > 0, \quad A_1, A_2, \ldots, A_n \in \Gamma$$

such that

$$g(t) = g_0 \, \exp(\tau_1 A_1) \, \exp(\tau_2 A_2) \cdots \exp(\tau_N A_N), \quad \tau_1 + \cdots + \tau_N = T.$$

(b) Define the *attainable set* (from $g \in \mathsf{G}$) $\mathcal{A}(g)$. Hence prove that

i. $\mathcal{A}(g) = \{g \, \exp(t_1 A_1) \cdots \exp(t_N A_N) \, : \, A_i \in \Gamma, \, t_i > 0, \, N \geq 0\}$.

ii. $\mathcal{A}(g) = g \, \mathcal{A}(\mathbf{1})$.

iii. $\mathcal{A}(\mathbf{1})$ is a subsemigroup of G.

iv. $\mathcal{A}(g)$ is a path-connected subset of G.

(c) If G is connected, what is the significance of the condition $\mathbf{1} \in \operatorname{int} \mathcal{A}(\mathbf{1})$? Make a clear statement (but DO NOT prove it).

[12,12,4]