RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 2 : OCTOBER 2011

MATHEMATICS HONOURS

GEOMETRIC CONTROL THEORY

AVAILABLE MARKS : 52
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**24** marks)

Let Z be a (real, finite-dimensional) vector space.

(a) Explain what is meant by saying that a map $\Omega : \mathsf{Z} \times \mathsf{Z} \to \mathbb{R}$ is a *symplectic form* on Z. Hence show that if Ω is a symplectic form on Z, then Z must be even-dimensional.

(b) Show that $\mathsf{Z} = \mathsf{W} \times \mathsf{W}^*$ admits a *canonical symplectic form*. (Here, W^* denotes the dual of the vector space W.)

(c) Verify that the formula for the symplectic form on $\mathbb{R}^{2n}$ as a matrix, namely

$$\mathbb{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(2n)\times(2n)}$$

coincides with the definition of the symplectic form as the canonical form on $\mathbb{R}^{2n}$ regarded as the product $\mathbb{R}^n \times (\mathbb{R}^n)^*$.

[12,6,6]

Question 2. (**28** marks)

Let (Z, Ω) be a (real, finite-dimensional) symplectic vector space.

(a) Explain what is meant by saying that a vector field $X : \mathsf{Z} \to \mathsf{Z}$ is *Hamiltonian.* Hence show that the Hamiltonian vector field X_H exists for any given (smooth) function $H : \mathsf{Z} \to \mathbb{R}$.

(b) Prove that a *linear* vector field $A : \mathsf{Z} \to \mathsf{Z}$ is Hamiltonian <u>if and only if</u> A is Ω-skew (i.e., $\Omega(Az_1, z_2) + \Omega(z_1, Az_2) = 0$ for all $z_1, z_2 \in \mathsf{Z}$).

(c) What about a general (smooth) vector field $X : \mathsf{Z} \to \mathsf{Z}$? Make a clear statement and then prove it.

[4,14,10]