RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : NOVEMBER 2009

MATHEMATICS HONOURS

Examiners : Dr C.C. Remsing
Dr F.A.M. Frescura

AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

GEOMETRIC CONTROL

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**20** marks]

Let Z be a finite-dimensional real vector space and let ω be a skew-symmetric bilinear form on Z.

(a) Explain what is meant by saying that ω is *nondegenerate*. Also, define the associated linear map

$$\omega^\flat : Z \to Z^*$$

(Z^* denotes the dual vector space).

(b) Prove that the following statements are equivalent :

i. ω is nondegenerate;

ii. the matrix $Q = \left[\omega(e_i, e_j)\right]$ of ω (with respect to a basis $(e_i)_{1 \leq i \leq m}$ of Z) is nonsingular;

iii. the linear map $\omega^\flat$ is an isomorphism.

(c) Define the term *symplectic vector space*. Hence, show that (the real vector space) $Z = W \times W^*$ admits a canonical symplectic structure.

[2,12,6]

Question 2. [**22** marks]

Let (Z, ω) be a symplectic vector space.

(a) Explain what is meant by saying that a vector field $X : Z \to Z$ is *Hamiltonian*. Hence, prove that a *linear* vector field $A : Z \to Z$ is Hamiltonian if and only if A is ω-skew (i.e.

$$\omega(Az_1,\, z_2) + \omega(z_1,\, Az_2) = 0$$

for all $z_1, z_2 \in Z$).

(b) Define the *Poisson bracket* $\{F, G\}$ of two functions $F, G \in C^\infty(Z)$. Hence, show that if $A, B : Z \to Z$ are linear Hamiltonian vector fields with corresponding energy functions

$$H_A(z) = \frac{1}{2}\,\omega(Az,\, z) \quad \text{and} \quad H_B(z) = \frac{1}{2}\,\omega(Bz,\, z),$$

then we have

$$\{H_A,\, H_B\} = H_{[A,B]}$$

($[A, B]$ denotes the Lie bracket (commutator) $A \circ B - B \circ A$).

[12,10]

Question 3. [**22** marks]

Let $\Sigma = (\mathsf{G}, \Gamma)$ be a *left-invariant* control system with the understanding that the state space G is a matrix Lie group and that the class $\mathcal{U}$ of admissible controls consists of *piecewise-constant* controls.

(a) Define the terms *trajectory* and *attainable set* (from $g \in \mathsf{G}$).

(b) Prove that

i. $\mathcal{A}(g) = \left\{ g\, e^{t_1 A_1} \cdots e^{t_N A_N} \mid A_i \in \Gamma, t_i > 0, N \geq 0 \right\}$.

ii. $\mathcal{A}(g) = g\,\mathcal{A}(1)$.

iii. $\mathcal{A}(1)$ is a sub-semigroup of G.

iv. $\mathcal{A}(g)$ is a path-connected subset of G.

[2, 20]

Question 4. [**22** marks]

Let $\Sigma = (\mathsf{G}, \Gamma)$ be a *left-invariant* control system with the understanding that the state space G is a matrix Lie group and that the class $\mathcal{U}$ of admissible controls consists of *piecewise-constant* controls. Let $\mathfrak{g}$ be the Lie algebra of G.

(a) For $\Gamma_1, \Gamma_2 \subseteq \mathfrak{g}$, we write $\Gamma_1 \sim \Gamma_2$ if $\mathrm{cl}\,\mathcal{A}_{\Gamma_1}(\mathbf{1}) = \mathrm{cl}\,\mathcal{A}_{\Gamma_2}(\mathbf{1})$. Show that

$$(\Gamma_1 \sim \Gamma \quad \text{and} \quad \Gamma_2 \sim \Gamma) \implies \Gamma_1 \cup \Gamma_2 \sim \Gamma.$$

(b) Define the *saturate* $\Sigma^{\mathrm{sat}} = (\mathsf{G}, \mathrm{Sat}\,(\Gamma))$ of Σ. Hence, prove that

i. $\mathrm{Sat}\,(\Gamma) \sim \Gamma$.

ii. $\mathrm{Sat}\,(\Gamma) = \{A \in \mathfrak{g} \mid \exp(\mathbb{R}_+ A) \subseteq \mathrm{cl}\,\mathcal{A}(\mathbf{1})\}$

($\mathrm{cl}\,\mathcal{A}_\Gamma(\mathbf{1})$ denotes the *topological closure* of the attainable set from the identity $\mathbf{1} \in \mathsf{G}$, corresponding to $\Gamma \subseteq \mathfrak{g}$.)

[6,16]

Question 5. [**24** marks]

Let G be a matrix Lie group with associated Lie algebra $\mathfrak{g}$.

(a) Define the *cotangent bundle* $T^*\mathsf{G}$, and then explain what is meant by the *left-invariant realization* of $T^*\mathsf{G}$.

(b) Explain the symplectic structure of the cotangent bundle, and then derive the left-invariant realization of the symplectic form $\omega = -d\theta$.

(c) Let $\vec{H} = (X, Y^*)$ denote the *Hamiltonian vector field* corresponding to the function H on $\mathsf{G} \times \mathfrak{g}^*$. Show that

$$\begin{aligned} X(g,p) &= \frac{\partial H}{\partial p}(g,p) \\ Y^*(g,p) &= -dL_g^*\left(\frac{\partial H}{\partial g}(g,p)\right) + \mathrm{ad}_X^*(p). \end{aligned}$$

[6,10,8]

END OF THE EXAMINATION PAPER