RHODES UNIVERSITY
DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : NOVEMBER 2011

MATHEMATICS HONOURS

Examiners : Dr C.C. Remsing
Prof. B. Makamba

AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

GEOMETRIC CONTROL THEORY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**18** marks]

Let (Z, Ω) be a (real, finite-dimensional) symplectic vector space, and let $C^\infty(\mathsf{Z})$ denote the *algebra* of smooth functions on Z.

(a) Define the *Poisson bracket* $\{F, G\}$ for $F, G \in C^\infty(\mathsf{Z})$, and then show that

$$X_{\{F,G\}} = -[X_F, X_G].$$

(b) Let X_H be a Hamiltonian vector field with *flow* φ_t. Show that

$$\frac{d}{dt}(F \circ \varphi_t) = \{F \circ \varphi_t,\, H\} = \{F, H\} \circ \varphi_t.$$

[10,8]

Question 2. [**20** marks]

(a) Define the *semi-Euclidean group* $\mathsf{SE}(1,1)$, and then show that $\mathsf{SE}(1,1)$ is a connected matrix Lie group.

(b) Determine the Lie algebra $\mathfrak{se}(1,1)$.

(c) Explain the terms *nilpotent Lie algebra* and *solvable Lie algebra*. Hence prove that the Lie algebra $\mathfrak{se}(1,1)$ is solvable but not nilpotent.

[8,4,8]

Question 3. [**18** marks]

State the *Campbell-Baker-Hausdorff Theorem* and then give ONLY ONE proof. (Provide as many details as you possibly can.)

[18]

Question 4. [**20** marks]

Let $\Sigma = (\mathsf{G}, \Gamma)$ be a left-invariant control system. (It is assumed that the state space G is a *matrix Lie group* and that the admissible controls are *piecewise-constant maps* $u(\cdot) : [0,T] \to U \subseteq \mathbb{R}^{\ell}$.)

(a) Explain what is meant by saying that a point $g' \in \mathsf{G}$ is *normally attainable* from a point $g \in \mathsf{G}$, and then state (but DO NOT PROVE) *Krener's Theorem*.

(b) Prove that if Σ has *full rank*, then $\text{int}_{\mathcal{O}}\mathcal{A} \neq \emptyset$. (Here $\mathcal{O}$ and $\mathcal{A}$ denote the orbit through, and the attainable set from, the identity $\mathbf{1} \in \mathsf{G}$, respectively.)

(c) Explain what is meant by saying that Σ is *controllable*, and then state and prove the *Group Test* (for controllability).

[4,8,8]

Question 5. [**16** marks]

Let G be a connected and simply connected matrix Lie group, and let $\Sigma = (\mathsf{G}, \Gamma)$ be a left-invariant control system (on G).

(a) Explain what is meant by saying that G is *connected* and *simply connected*.

(b) Assume that Γ is contained in a half-space in (the Lie algebra) $\mathfrak{g}$ bounded by a Lie subalgebra. Prove that Σ is <u>not</u> controllable.

[2,14]

Question 6. [**18** marks]

Discuss ONLY ONE of the following problems : the *sub-Riemannian problem* (on the Heisenberg group H_3), the *elastic problem* (on the Euclidean group $\mathsf{SE}(2)$) or the *plate-ball problem* (on the group $\mathsf{SO}(3) \times \mathbb{R}^2$). (Provide as many details as you possibly can.)

[18]

END OF THE EXAMINATION PAPER