Key Results

Geometry: Naive Lie Theory (Honours)

C.C. Remsing
Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown 6140

(Geometry of complex numbers and quaternions)

- (p.14): **Rotation by conjugation.** *If* $t = \cos\theta + u\sin\theta$, *where* $u \in \mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ *is a unit vector, then conjugation by* t *rotates* $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ *through angle* 2θ *about axis* u.

- (p.16): **Rotations form a group.** *The product of rotations is a rotation, and the inverse of a rotation is a rotation.*

(Groups)

- (p.26): $\mathbb{S}^3$ *can be decomposed into disjoint congruent circles.*

- (p.33): **Simplicity of SO(3).** *The only nontrivial subgroup of* $\mathsf{SO}(3)$ *closed under conjugation is* $\mathsf{SO}(3)$ *itself.*

- (p.37): **Reflection representation of isometries.** *Any isometry of* $\mathbb{R}^n$ *that fixes* O *is the product of at most* n *reflections in hyperplanes through* O.

- (p.38): **Quaternion representation of reflections.** *Reflection of* $\mathbb{H} = \mathbb{R}^4$ *in the hyperplane through* O *orthogonal to the unit quaternion* u *is the map that sends each* $q \in \mathbb{H}$ *to* $-u\bar{q}u$.

- (p.39): **Quaternion representation of rotations.** *Any rotation of* $\mathbb{H} = \mathbb{R}^4$ *about* O *is a map of the form* $q \mapsto vqw$, *where* v *and* w *are unit quaternions.*

- (p.43): **Size of the kernel.** *The homomorphism* $\varphi : \mathsf{SU}(2)\times\mathsf{SU}(2) \to \mathsf{SO}(4)$ *is 2-to-1, because its kernel has two elements.*

- (p.44): **SO(4) is not simple.** *There is a nontrivial normal subgroup of* $\mathsf{SO}(4)$, *not equal to* $\mathsf{SO}(4)$.

(Generalized rotation groups)

- (p.50): **Rotation criterion.** *An* $n \times n$ *real matrix* A *represents a rotation of* $\mathbb{R}^n$ *if and only if* $AA^\top = \mathbf{1}$ *and* $\det(A) = 1$.

- (p.53): **Path-connectedness of SO(n).** *For any* n, $\mathsf{SO}(n)$ *is path-connected.*

- (p.55): **Criterion for preserving the inner product on** $\mathbb{C}^n$**.** *A linear transformation of* $\mathbb{C}^n$ *preserves the inner product*

$$(u_1, u_2, \ldots, u_n) \bullet (v_1, v_2, \ldots, v_n) = u_1\overline{v}_1 + u_2\overline{v}_2 + \cdots + u_n\overline{v}_n$$

if and only if its matrix A *satisfies* $A\overline{A}^\top = \mathbf{1}$, *where* $\mathbf{1}$ *is the identity matrix.*

- (p.64): **Maximal tori in generalized rotation groups.** *The tori listed above are maximal in the corresponding groups.*

- (p.67): **Centers of generalized rotation groups.** *The centers of the groups* $\mathsf{SO}(n)$, $\mathsf{U}(n)$, $\mathsf{SU}(n)$, $\mathsf{Sp}(n)$ *are:*

  1. $Z(\mathsf{SO}(2m)) = \{\pm \mathbf{1}\}$.
  2. $Z(\mathsf{SO}(2m+1)) = \{\mathbf{1}\}$.
  3. $Z(\mathsf{U}(n)) = \{\omega\mathbf{1} : |\omega| = 1\}$.
  4. $Z(\mathsf{SU}(n)) = \{\omega\mathbf{1} : \omega^n = 1\}$.
  5. $Z(\mathsf{Sp}(n)) = \{\pm \mathbf{1}\}$.

- (p.69): **Centrality of discrete normal subgroups.** *If* G *is a path-connected matrix Lie group with a discrete normal subgroup* H, *then* H *is contained in the center* $Z(\mathsf{G})$ *of* G.

(The exponential map)

- (p.77): **Exponentiation theorem for $\mathbb{H}$.** *When we write an arbitrary element of* $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ *in the form* $\theta\, u$, *where* u *is a unit vector, we have* $e^{\theta u}=\cos\theta+u\,\sin\theta$ *and the exponential function maps* $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ *onto* $\mathbb{S}^3=\mathsf{SU}(2)$.

- (p.84): **Submultiplicative property.** *For any two real* $n\times n$ *matrices* A *and* B, $|AB|\leq|A|\,|B|$.

- (p.85): **Convergence of the exponential series.** *If* A *is any* $n\times n$ *real matrix, then* $\mathbf{1}+\frac{A}{1!}+\frac{A^2}{2!}+\frac{A^3}{3!}+\cdots$ *is convergent in* $\mathbb{R}^{n^2}$.

(The tangent space)

- (p.95): **Tangent vectors of O(n), U(n), Sp(n).** *The tangent vectors* X *at* $\mathbf{1}$ *are matrices of the following forms:*

  (a) For $\mathsf{O}(n)$, $n\times n$ *real matrices* X *such that* $X+X^\top=\mathbf{0}$.

  (b) For $\mathsf{U}(n)$, $n\times n$ *complex matrices* X *such that* $X+\overline{X}^\top=\mathbf{0}$.

  (c) For $\mathsf{Sp}(n)$, $n\times n$ *quaternion matrices* X *such that* $X+\overline{X}^\top=\mathbf{0}$.

- (p.97): **Tangent space of SO(n).** *The tangent space of* $\mathsf{SO}(n)$ *consists of precisely the* $n\times n$ *real vectors* X *such that* $X+X^\top=\mathbf{1}$.

- (p.99): **Tangent space of U(n) and Sp(n).** *The tangent space of* $\mathsf{U}(n)$ *consists of all the* $n\times n$ *complex matrices satisfying* $X+\overline{X}^\top=\mathbf{0}$. *The tangent space of* $\mathsf{Sp}(n)$ *consists of all* $n\times n$ *quaternion matrices* X *satisfying* $X+\overline{X}^\top=\mathbf{0}$, *where* $\overline{X}$ *denotes the quaternion conjugate of* X.

- (p.100): **Determinant of exp.** *For any square matrix* A, $\det\left(e^A\right)=e^{\mathrm{Tr}\,(A)}$.

- (p.101): **Tangent space of SU(n).** *The tangent space of* $\mathsf{SU}(n)$ *consists of all* $n\times n$ *complex matrices* X *such that* $X+\overline{X}^\top=\mathbf{0}$ *and* $\mathrm{Tr}\,(X)=0$.

- (p.103): **Vector space properties.** *$T_{\mathbf{1}}(\mathsf{G})$ is a vector space over $\mathbb{R}$; that is, for any $X, Y \in T_{\mathbf{1}}(\mathsf{G})$ we have $X+Y \in T_{\mathbf{1}}(\mathsf{G})$ and $rX \in T_{\mathbf{1}}(\mathsf{G})$ for any real r.*

- (p.104): **Lie bracket property.** *$T_{\mathbf{1}}(\mathsf{G})$ is closed under the Lie bracket, that is, if $X, Y \in T_{\mathbf{1}}(\mathsf{G})$ then $[X, Y] \in T_{\mathbf{1}}(\mathsf{G})$, where $[X, Y] = XY - YX$.*

- (p.106): **Dimension of so(n), u(n), su(n), and sp(n).** *As vector spaces over $\mathbb{R}$,*

  (a) $\mathfrak{so}(n)$ has dimension $n(n-1)/2$.
  (b) $\mathfrak{u}(n)$ has dimension n^2.
  (c) $\mathfrak{su}(n)$ has dimension $n^2 - 1$.
  (d) $\mathfrak{sp}(n)$ has dimension $n(2n+1)$.

(Structure of Lie algebras)

- (p.117): **Tangent space of a normal subgroup.** *If H is a normal subgroup of a matrix Lie group G, then $T_{\mathbf{1}}(\mathsf{H})$ is an ideal of the Lie algebar $T_{\mathbf{1}}(\mathsf{G})$.*

- (p.119): **Simplicity of the cross-product algebra.** *The cross-product algebra is simple.*

- (p.120): **Kernel of a Lie algebra homomorphism.** *If $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism, and $\mathfrak{h} = \{X \in \mathfrak{g} : \varphi(X) = \mathbf{0}\}$ is its kernel, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

- (p.125): **Simplicity of sl(n,C).** *For each n, $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra.*

- (p.126): **Simplicity of su(n).** *For each n, $\mathfrak{su}(n)$ is a simple Lie algebra.*

- (p.130): **Simplicity of so(n).** *For each $n > 4$, $\mathfrak{so}(n)$ is a simple Lie algebra*

- (p.134): **Simplicity of sp(n).** *For all n, $\mathfrak{sp}(n)$ is a simple Lie algebra.*

(The matrix logarithm)

- (p.140): **Inverse property of matrix logarithm.** *For any matrix* e^X *within distance* 1 *of the identity,* $\log\left(e^X\right) = X$.

- (p.141): **Multiplicative property of matrix logarithm.** *If* $AB = BA$*, and* $\log(A)$, $\log(B)$*, and* $\log(AB)$ *are all defined, then* $\log(AB) = \log(A) + \log(B)$.

- (p.143): **Exponentiation of tangent vectors.** *If* $A'(0)$ *is the tangent vector at* $\mathbf{1}$ *to a matrix Lie group* G*, then* $e^{A'(0)} \in \mathsf{G}$*. That is,* $\exp$ *maps the tangent space* $T_{\mathbf{1}}(\mathsf{G})$ *into* G.

- (p.146): **Smoothness of sequential tangency.** *Suppose that* (A_m) *is a sequence in a matrix Lie group* G *such that* $A_m \to \mathbf{1}$ *as* $m \to \infty$*, and that* (α_m) *is a sequence of real numbers such that* $(A_m - \mathbf{1})/\alpha_m \to X$ *as* $m \to \infty$*. Then* $e^{tX} \in \mathsf{G}$ *for all real* t *(and therefore* X *is the tangent at* $\mathbf{1}$ *to the smooth path* e^{tX}*).*

- (p.148): **The log of a neighborhood of 1.** *For any matrix Lie group* G *there is a neighborhood* $N_\delta(\mathbf{1})$ *mapped into* $T_{\mathbf{1}}(\mathsf{G})$ *by* $\log$.

- (p.149): **Corollary.** *The* $\log$ *function gives a bijection, continuous in both directions, between* $N_\delta(\mathbf{1})$ *in* G *and* $\log N_\delta(\mathbf{1})$ *in* $T_{\mathbf{1}}(\mathsf{G})$.

- (p.150): **Tangent space visibility.** *If* G *is a path-connected matrix Lie group with discrete center and a nondiscrete normal subgroup* H*, then* $T_{\mathbf{1}}(\mathsf{H}) \neq \{\mathbf{0}\}$.

- (p.151): **Corollary.** *If* H *is a nontrivial normal subgroup of* G *under the conditions above, then* $T_{\mathbf{1}}(\mathsf{H})$ *is a nontrivial ideal of* $T_{\mathbf{1}}(\mathsf{G})$.

- (p.154): **Campbell-Baker-Hausdorff theorem.** *For each* $n \geq 1$*, the polynomial* $F_n(A, B)$ *in*

$$e^A e^B = e^Z, \quad Z = F_1(A,B) + F_2(A,B) + F_3(A,B) + \cdots$$

is Lie.

(Topology)

- (p.169): **Heine-Borel theorem.** *If* $[0,1]$ *is contained in a union of open intervals* $\mathcal{U}_i$, *then the union of finitely many* $\mathcal{U}_i$ *also contains* $[0,1]$.

- (p.171): **Continuous image of a compact set.** *If* $\mathcal{K}$ *is compact and* f *is a continuous function defined on* $\mathcal{K}$, *then* $f(\mathcal{K})$ *is compact.*

- (p.172): **Uniform continuity.** *If* $\mathcal{K}$ *is a compact subset of* $\mathbb{R}^m$ *and* $f : \mathcal{K} \to \mathbb{R}^n$ *is continuous, then* f *is uniformly continuous.*

- (p.175): **Normality of the identity component.** *If* G^0 *is the identity component of a matrix Lie group* G, *then* G^0 *is a normal subgroup of* G.

- (p.176): **Generating a path-connected group.** *If* G *is a path-connected matrix Lie group, and* $N_\delta(\mathbf{1})$ *is a neighborhood of* $\mathbf{1}$ *in* G, *then any element of* G *is a product of members of* $N_\delta(\mathbf{1})$.

- (p.177): **Corollary.** *If* G *is a path-connected matrix Lie group, then each element of* G *has the form* $e^{X_1}e^{X_2}\cdots e^{X_m}$ *for some* $X_1, X_2, \ldots, X_m \in T_{\mathbf{1}}(\mathsf{G})$.

- (p.179): **Unique path lifting.** *Suppose that* p *is a path in* $\mathbb{S}^1$ *with initial point* P, *and* $\tilde{P}$ *is a point in* $\mathbb{R}$ *over* Q. *Then there is a unique path* $\tilde{p}$ *in* $\mathbb{R}$ *such that* $\tilde{p}(0) = \tilde{P}$ *and* $f \circ \tilde{p} = p$. *We call* $\tilde{p}$ *the lift of* p *with initial point* $\tilde{P}$.

(Simply connected Lie groups)

- (p.191): **The induced homomorphism.** *For any Lie homomorphism* $\Phi : \mathsf{G} \to \mathsf{H}$ *of matrix Lie groups* G, H, *with Lie algebras* $\mathfrak{g}, \mathfrak{h}$, *respectively, there is a Lie homomorphism* $\varphi : \mathfrak{g} \to \mathfrak{h}$ *such that* $\varphi(A'(0)) = (\phi \circ A)'(0)$ *for any smooth path* $A(t)$ *through* $\mathbf{1}$ *in* G.

- (p.194): **Uniform continuity of paths.** *If* $p : [0,1] \to \mathbb{R}^n$ *is a path, then, for any* $\varepsilon > 0$, *it is possible to devide* $[0,1]$ *into a finite number of subintervals, each of which is mapped by* p *into an open ball of radius* ε.

- (p.194): **Uniform continuity of path deformations.** *If* $d : [0,1] \times [0,1] \to \mathbb{R}^n$ *is a path deformation, then, for any* $\varepsilon > 0$, *it is possible to devide the square* $[0,1] \times [0,1]$ *into a finite number of subsquares, each of which is mapped by* d *into an open ball of radius* ε.

- (p.201): **Homomorphisms of simply connected groups.** *If* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are the Lie algebras of the simply connected Lie groups* G *and* H, *respectively, and if* $\varphi : \mathfrak{g} \to \mathfrak{h}$ *is a homomorphism, then there is a homomorphism* $\Phi : \mathsf{G} \to \mathsf{H}$ *that induces* φ.

- (p.201): **Corollary.** *If* G *and* H *are simply connected Lie groups with isomorphic Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$, *respectively, then* G *is isomorphic to* H.