RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 1 : MARCH 2010

MATHEMATICS HONOURS

GEOMETRY (NAIVE LIE THEORY)

AVAILABLE MARKS : 54
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**27** marks)

(a) Define the term *quaternion*, and then explain what is meant by saying that the set $\mathbb{H}$ of all quaternions has an *algebra* structure.

(b) Given a quaternion $q = a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, define the *quaternion conjugate* $\overline{q}$, and then show that

$$\overline{q_1 q_2} = \overline{q}_2\,\overline{q}_1.$$

(c) Prove that the set $\mathbb{S}^3$ of unit quaternions is a *group* under the quaternion multiplication.

(d) Let

$$t = \cos\theta + u\,\sin\theta, \quad u \in \mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k}$$

be a unit quaternion. Show that *conjugation* by t rotates $\mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k}$ through angle 2θ about axis u.

[3,4,5,15]

Question 2. (**27** marks)

(a) Prove that any *isometry* of $\mathbb{R}^n$ that fixes O is the product of at most n reflections in hyperplanes through O.

(b) Show that *reflection* in the hyperplane orthogonal to a coordinate axis has determinant -1, and generalize this result to any reflection.

(c) Prove that any *rotation* of $\mathbb{H} = \mathbb{R}^4$ about O is a map of the form

$$q \mapsto v\, q\, w$$

where v and w are unit quaternions.

[10,5,12]