RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 2 : APRIL 2010

MATHEMATICS HONOURS

GEOMETRY (NAIVE LIE THEORY)

AVAILABLE MARKS : 54
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**27** marks)

(a) Define the *absolute value* and the *exponential* of a (complex) $n \times n$ matrix $A = (a_{ij})$, and then prove the *submultiplicative property*:

$$|AB| \leq |A|\,|B|\,.$$

(b) If A is a matrix of the form BCB^{-1}, show that

$$e^A = B\,e^C\,B^{-1}.$$

(c) Define the *affine group* $\mathsf{Aff}\,(1)$, and then determine its Lie algebra $\mathfrak{aff}\,(1)$. Hence, show that the exponential map

$$\exp : \mathfrak{aff}\,(1) \to \mathsf{Aff}\,(1), \quad A \mapsto e^A$$

is *surjective*.

[7,6,14]

Question 2. (**27** marks)

(a) Define the matrix groups $\mathsf{U}(n)$ and $\mathsf{SU}(n)$, and then explain what is meant by saying that (the matrix) X is a *tangent vector* of $\mathsf{U}(n)$ at (the identity) $\mathbf{1}$. Hence, determine the form of such a tangent vector.

(b) Prove that the *tangent space* $T_{\mathbf{1}}\mathsf{SU}(n)$ consists of precisely the $n \times n$ complex matrices X such that

$$X + \overline{X}^{\top} = \mathbf{0} \quad \text{and} \quad \mathrm{tr}(X) = 0.$$

(c) Consider the determinant map

$$\det : \mathsf{U}(n) \to \mathbb{C}.$$

Why is this a *homomorphism* ? What is its *kernel* ? Hence, deduce that $\mathsf{SU}(n)$ is a *normal subgroup* of $\mathsf{U}(n)$.

[7,10,10]