RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 1 : APRIL 2011

MATHEMATICS HONOURS

GEOMETRY (NAIVE LIE THEORY)

AVAILABLE MARKS : 53
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**23** marks)

(a) Define the term *quaternion*, and then explain what is meant by saying that the set $\mathbb{H}$ of all quaternions has an *algebra* structure.

(b) Given pure quaternions $p, q \in \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$, define the *vector product*

$$p \times q = (p_2q_3 - p_3q_2)\mathbf{i} + (p_3q_1 - p_1q_3)\mathbf{j} + (p_1q_2 - p_2q_1)\mathbf{k}.$$

Prove that (for pure imaginary quaternions p, q, r)

$$p \times (q \times r) = q\,(p \bullet r) - r\,(p \bullet q).$$

(c) Hence, deduce the *Jacobi identity*

$$p \times (q \times r) + q \times (r \times p) + r \times (p \times q) = \mathbf{0}.$$

(d) Show that the *reflection* of $\mathbb{H} = \mathbb{R}^4$ in the hyperplane through the origin O orthogonal to the unit quaternion u is the map

$$q \in \mathbb{H} \mapsto -u\,\bar{q}\,u \in \mathbb{H}.$$

[3,6,2,12]

Question 2. (**30** marks)

(a) Prove that an $n \times n$ real matrix A represents a *rotation* of (the Euclidean space) $\mathbb{R}^n$, $n \geq 2$ <u>if and only if</u>

$$A\,A^\top = \mathbf{1} \quad \text{and} \quad \det(A) = 1.$$

(b) Explain what is meant by a *maximal torus* of (the matrix group) $\mathsf{SO}(n)$ and then identify a maximal torus in $\mathsf{SO}(4)$. Prove your assertion.

(c) Explain what is meant by the *center* $Z(\mathsf{SO}(n))$ of $\mathsf{SO}(n)$, and then prove that

$$Z(\mathsf{SO}(4)) = \{-\mathbf{1}, \mathbf{1}\}\,.$$

[10,12,8]