RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 2 : MAY 2011

MATHEMATICS HONOURS

GEOMETRY (NAIVE LIE THEORY)

AVAILABLE MARKS : 52
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**26** marks)

(a) Given a matrix group G, define its *tangent space at the identity*, $T_{\mathbf{1}}\mathsf{G}$. Hence, prove that

   i. $T_{\mathbf{1}}\mathsf{G}$ is a vector space (over $\mathbb{R}$);

   ii. $T_{\mathbf{1}}\mathsf{G}$ is closed under the Lie bracket (matrix commutator).

(b) Prove that (the tangent space of the rotation group $\mathsf{SO}(n)$ at the identity) $T_{\mathbf{1}}\mathsf{SO}(n)$ consists of precisely the $n \times n$ real matrices X such that $X + X^{\top} = \mathbf{0}$.

[6,12,8]

Question 2. (**26** marks)

(a) Define the terms *Lie algebra* and *ideal* (in a Lie algebra) and then explain what is meant by saying that a Lie algebra is *simple.* Hence, show that the cross-product Lie algebra $\mathbb{R}^3_{\wedge}$ is simple.

(b) Show that $\mathrm{Tr}([X, Y]) = 0$ for any $X, Y \in \mathfrak{gl}(n, \mathbb{C})$. Hence, deduce that $\mathfrak{sl}(n, \mathbb{C})$ is an ideal of $\mathfrak{gl}(n, \mathbb{C})$.

[16,10]