RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 1 : APRIL 2012

MATHEMATICS HONOURS

GEOMETRY (NAIVE LIE THEORY)

AVAILABLE MARKS : 54
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**26** marks)

(a) Define the term *quaternion*, and then show that the product of quaternions is a quaternion.

(b) Given a quaternion q, define the *quaternion conjugate* $\overline{q}$ and the *absolute value* $|q|$. Hence show that $q\overline{q} = |q|^2$.

(c) Let u be a pure imaginary quaternion with $|u| = 1$. Prove that $u^2 = -1$.

(d) Explain what is meant by saying that a quaternion t of absolute value 1 can be expressed as

$$t = \cos\theta + u\,\sin\theta$$

where u is a unit vector in $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$. Hence prove that the conjugation by t rotates $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ through angle 2θ about axis u.

[4,6,4,12]

Question 2. (**28** marks)

(a) Prove that the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ is a group. Is this group Abelian? Justify your answer.

(b) Prove that $\mathbb{S}^3$ can be decomposed into disjoint congruent circles.

(c) Define the term *direct product of groups*. Hence show that $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is a group.

(d) Explain what is meant by saying that two groups are *isomorphic*. Hence prove that the groups $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ are <u>not</u> isomorphic.

[6,10,6,6]