RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 2 : MAY 2012

MATHEMATICS HONOURS

GEOMETRY (NAIVE LIE THEORY)

AVAILABLE MARKS : 54
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**26** marks)

(a) Define the matrix group $\mathsf{SU}(2)$ and show that $\mathsf{SU}(2) = \mathbb{S}^3 \subset \mathbb{R}^4$.

(b) Define the *tangent space* at $\mathbf{1}$ of $\mathsf{SU}(2)$. Hence show that the tangent space of $\mathsf{SU}(2)$ at $\mathbf{1}$ is $\mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k}$.

(c) Let $T_{\mathbf{1}}\mathsf{SU}(2) = \mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k}$. Prove that if $U, V \in T_{\mathbf{1}}\mathsf{SU}(2)$, then (the matrix commutator) $[U, V] \in T_{\mathbf{1}}\mathsf{SU}(2)$.

[6,8,12]

Question 2. (**28** marks)

(a) Define the term *matrix Lie algebra* and then carefully explain what is meant by saying that any real matrix Lie algebra $\mathfrak{g}$ has a *complexification* $\mathfrak{g} + i\,\mathfrak{g}$.

(b) Define the (matrix) Lie algebra $\mathfrak{su}(n)$ and then show that $\mathfrak{su}(n)$ is <u>not</u> a *vector space* over $\mathbb{C}$.

(c) Prove that the complexification of $\mathfrak{su}(n)$ is $\mathfrak{sl}(n, \mathbb{C})$.

[6,10,12]