RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 1 : MARCH 2013

MATHEMATICS HONOURS

GEOMETRY (NAIVE LIE THEORY)

AVAILABLE MARKS : 55

FULL MARKS : 50

DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. (**25** marks)

(a) Given a quaternion $q = a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \in \mathbb{H}$, define the *quaternion conjugate* $\overline{q}$ and then show that

$$\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1.$$

(b) Prove that the set $\mathbb{S}^3 \subset \mathbb{H} = \mathbb{R}^4$ of unit quaternions forms a *group*, under the quaternion multiplication.

(c) Explain what is meant by saying that a quaternion t of absolute value 1 can be expressed as

$$t = \cos\theta + u \sin\theta$$

where u is a unit vector in $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$. Hence prove that the conjugation by t rotates $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ through angle 2θ about axis u.

[5,5,15]

Question 2. (**30** marks)

(a) Define what is meant by a *rotation* in (the Euclidean space) $\mathbb{R}^n$. Hence prove that an $n \times n$ real matrix A represents a rotation if and only if

$$AA^\top = \mathbf{1} \quad \text{and} \quad \det A = 1.$$

(b) Explain what is meant by saying that a set $\mathsf{S} \subset \mathbb{R}^n$ is *path-connected.* Hence prove that (the rotation group)

$$\mathsf{SO}\,(3) \subset \mathsf{Mat}\,(3, \mathbb{R}) = \mathbb{R}^9$$

is path-connected.

(c) Define the *center* $Z(\mathsf{G})$ of the (matrix Lie) group G, and then compute $Z(\mathsf{SO}\,(3))$.

[10,10,10]