RHODES UNIVERSITY
DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : MAY 2010

MATHEMATICS HONOURS

Examiners : Dr C.C. Remsing
Prof B. Makamba

AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

GEOMETRY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**16** marks]

(a) Define the groups $\mathsf{SO}\,(3)$, $\mathsf{SU}\,(2)$ and $\mathsf{SO}\,(4)$. Are the groups $\mathsf{SO}\,(3)$ and $\mathsf{SU}\,(2)$ *simple* ? Make clear statements but DO NOT prove them.

(b) Explain why $\mathbb{S}^3 = \mathsf{SU}\,(2)$ is *not* the same group as $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

(c) Consider the map

$$\varphi : \mathsf{SU}\,(2) \times \mathsf{SU}\,(2) \to \mathsf{SO}\,(4), \quad (v, w) \mapsto \varphi(v, w) : q \mapsto v^{-1}qw.$$

(Recall that any rotation of $\mathbb{H} = \mathbb{R}^4$ about the origin is a map of the form $q \mapsto vqw$, where v and w are unit quaternions.) Show that

i. φ is a group *homomorphism.*

ii. the kernel of φ has two elements.

(d) Prove that $\mathsf{SO}\,(4)$ is *not* simple. [Hint : Show that there is a nontrivial normal subgroup of $\mathsf{SO}\,(4)$, not equal to $\mathsf{SO}\,(4)$.]

[3,3,6,4]

Question 2. [**18** marks]

(a) Define the special unitary group $\mathsf{SU}(n)$, and then prove that $\mathsf{SU}(n)$ is *path-connected*.

(b) Find the *centre* of $\mathsf{SU}(n)$.

(c) Is $\mathsf{SU}(2)/Z(\mathsf{SU}(2)) = \mathsf{SO}(3)$? Justify your answer.

[6,8,4]

Question 3. [**16** marks]

Let $A, B \in \mathbb{C}^{n\times n}$.

(a) Define the *matrix exponential* e^A, and then show (by term-by-term differentiation, or otherwise) that

$$\frac{d}{dt}\, e^{tA} = A\, e^{tA} = e^{tA}\, A.$$

(b) Prove that

$$\det\left(e^A\right) = e^{\operatorname{tr} A}.$$

(c) Show that

$$A^\top e^B A = e^{A^\top BA}$$

for any *orthogonal* matrix A.

[4,8,4]

Question 4. [**22** marks]

(a) Given a matrix group G, define the *tangent space* (at the identity) $T_{\mathbf{1}}\mathsf{G}$ and then prove that

i. $T_{\mathbf{1}}\mathsf{G}$ is a *vector space* (over $\mathbb{R}$).

ii. $T_{\mathbf{1}}\mathsf{G}$ is closed under the Lie bracket.

(b) Show that each element of the tangent space of $\mathsf{SO}(3)$ has the form

$$X = \begin{bmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{bmatrix} = x\,\mathbf{I} + y\,\mathbf{J} + z\,\mathbf{K}$$

where

$$\mathbf{I} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{J} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Deduce that $T_{\mathbf{1}}\mathsf{SO}(3)$ is a real *vector space* of dimension 3.

(c) Check that

$$[\mathbf{I}, \mathbf{J}] = \mathbf{K}, \quad [\mathbf{J}, \mathbf{K}] = \mathbf{I}, \quad [\mathbf{K}, \mathbf{I}] = \mathbf{J}.$$

Hence, deduce that the Lie algebra $T_{\mathbf{1}}\mathsf{SO}(3)$ is *isomorphic* to (the Lie algebra) $\mathbf{R}^3$ under the cross product operation.

[12,5,5]

Question 5. [**20** marks]

(a) Define the term *ideal* (of a Lie algebra), and then show that if $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a *Lie algebra homomorphism*, then its kernel is an ideal.

(b) Show that $\operatorname{tr}([X, Y]) = 0$ for any $X, Y \in \mathfrak{gl}(n, \mathbb{C})$. Hence, deduce that $\mathfrak{sl}(n, \mathbb{C})$ is an *ideal* of $\mathfrak{gl}(n, \mathbb{C})$.

(c) Explain what is meant by saying that a Lie algebra is *simple*, and then prove that $\mathfrak{so}(n)$, $n > 4$ is simple.

[4,4,12]

Question 6. [**18** marks]

(a) Define the *matrix logarithm*, and then prove that, for any matrix e^X within distance 1 from the identity,

$$\log\left(e^X\right) = X.$$

(b) Define the term *matrix Lie group*, and then prove that if $A'(0)$ is a tangent vector at $\mathbf{1}$ to a matrix Lie group G, then $e^{A'(0)} \in \mathsf{G}$ (i.e., exp maps the tangent space $T_{\mathbf{1}}\mathsf{G}$ into G).

(c) Show that if G is a path-connected matrix Lie group with discrete centre and a nondiscrete normal subgroup H, then $T_{\mathbf{1}}\mathsf{H} \neq \{\mathbf{0}\}$.

[3,8,7]

END OF THE EXAMINATION PAPER