RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2011

MATHEMATICS HONOURS

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AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

GEOMETRY (LIE THEORY)

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**16** marks]

(a) Show (by using quaternions or otherwise) that the 3-sphere

$$\mathbb{S}^3 = \left\{(x, y, z, t) \in \mathbb{R}^4 \, : \, x^2 + y^2 + z^2 + t^2 = 1\right\} \subset \mathbb{R}^4$$

can be regarded as a matrix Lie group.

(b) Define the *orthogonal group* $\mathsf{SO}(3)$ and then show that the unit circle $\mathbb{S}^1$ is <u>not</u> a *normal subgroup* of $\mathsf{SO}(3)$.

(c) Give a direct, geometric proof of the fact that the only nontrivial normal subgroup of $\mathsf{SO}(3)$ is $\mathsf{SO}(3)$ itself.

[3,5,8]

Question 2. [**16** marks]

(a) Explain what is meant by the *Hermitian inner product* on the complex vector space $\mathbb{C}^n$. Hence, define the *unitary groups* $\mathsf{U}(n)$ and $\mathsf{SU}(n)$.

(b) Prove that a linear transformation on $\mathbb{C}^n$ preserves the Hermitian inner product if and only if its matrix A satisfies

$$A\bar{A}^{\top} = \mathbf{1}.$$

(c) Define the terms *maximal torus* and *center* of a matrix Lie group. What is the maximal torus in $\mathsf{U}(n)$? (Make a clear statement but DO NOT prove it.) Find the center of $\mathsf{U}(n)$.

[2,6,8]

Question 3. [**18** marks]

(a) Define the *matrix exponential* e^A, and then show (by term-by-term differentiation, or otherwise) that

$$\frac{d}{dt}\, e^{tA} = A\, e^{tA} = e^{tA}\, A.$$

(b) Define the *tangent space* at (the identity) $\mathbf{1}$ of a matrix Lie group G, $T_{\mathbf{1}}(\mathsf{G})$. Hence, find $T_{\mathbf{1}}(\mathsf{SL}(2,\mathbb{C}))$. Verify that $T_{\mathbf{1}}(\mathsf{SL}(2,\mathbb{C}))$ is (real) Lie algebra of dimension three.

(c) Show that the matrix $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ in $\mathsf{SL}(2,\mathbb{C})$ is not equal to e^X for any $X \in T_1(\mathsf{SL}(2,\mathbb{C}))$. (Hint : Use the *Cayley-Hamilton* theorem to show that $X^2 = -\det(X)\,\mathbf{1}$ whenever $\operatorname{Tr}(X) = 0$. Then show that $e^X = \cos(\sqrt{\det(X)}\mathbf{1} + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}} X$ and hence derive a contradiction.)

[4,6,8]

Question 4. [**20** marks]

(a) Define the term *ideal* (of a Lie algebra), and then explain what is meant by saying that a Lie algebra is *simple*. Hence, prove that $\mathfrak{sl}(n, \mathbb{C})$ is simple.

(b) Consider the Lie algebra $\mathfrak{gl}(2, \mathbb{H})$ and the sets

$$\begin{aligned} \mathfrak{R} &= \{X \in \mathfrak{gl}(2, \mathbb{H}) \,:\, X = r\,\mathbf{1},\ r \in \mathbb{R}\} \\ \mathfrak{T} &= \{X \in \mathfrak{gl}(2, \mathbb{H}) \,:\, \mathrm{re}\,(\mathrm{Tr}\,(X)) = 0\} \end{aligned}$$

where re denotes the real part of the quaternion.

i. Prove that $\mathfrak{R}$ and $\mathfrak{T}$ are *ideals* of $\mathfrak{gl}(2, \mathbb{H})$.

ii. Show that each element $X \in \mathfrak{gl}(2, \mathbb{H})$ has a *unique decomposition* of the form

$$X = R + T$$

where $R \in \mathfrak{R}$ and $T \in \mathfrak{T}$.

[10,10]

Question 5. [**20** marks]

(a) Define the *matrix logarithm*, and then prove that, if $AB = BA$ and $\log(A)$, $\log(B)$ and $\log(AB)$ are all defined, then

$$\log(AB) = \log(A) + \log(B).$$

(b) Define the term *matrix Lie group*, and then prove that for any matrix Lie group G, there is a neighborhood $N_\delta(\mathbf{1})$ mapped into (the tangent space) $T_{\mathbf{1}}(\mathsf{G})$ by log.

(c) Prove that (the special unitary group) $\mathsf{SU}(n)$ is a *normal* subgroup of (the unitary group) $\mathsf{U}(n)$.

[5,10,5]

Question 6. [**20** marks]

(a) Explain what is meant by saying that a matrix Lie group G is *path-connected*. Hence, prove that if G is a path-connected Lie group and $N_\delta(\mathbf{1})$ is a neighborhood of $\mathbf{1}$ in G, then any element of G is a product of members of $N_\delta(\mathbf{1})$.

(b) Show that

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = e^X e^Y$$

for some $X, Y \in T_{\mathbf{1}}(\mathsf{SL}(2,\mathbb{C}))$. (HINT: Write $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ as the product of two matrices in $\mathsf{SL}(2,\mathbb{C})$ with entries 0, i or $-i$.)

(c) Define carefully the term *Lie group homomorphism*. Hence, prove that for any Lie group homomorphism $\Phi : \mathsf{G} \to \mathsf{H}$ (of matrix Lie groups G and H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively), there is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\phi(A'(0)) = (\Phi \circ A)'(0)$$

for any smooth path $A(\cdot)$ through $\mathbf{1}$ in G.

[8,6,6]

END OF THE EXAMINATION PAPER