RHODES UNIVERSITY
DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2012

MATHEMATICS HONOURS

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AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

GEOMETRY (LIE THEORY)

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**20** marks]

(a) Explain what is meant by a *rotation* in $\mathbb{R}^n$. Hence prove that an $n \times n$ real matrix A represents a rotation of $\mathbb{R}^n$ <u>if and only if</u>

$$AA^\top = \mathbf{1} \quad \text{and} \quad \det(A) = 1.$$

(b) Define a *maximal torus* and the *center* of a (matrix) Lie group. Hence give

- (<u>without</u> justification) a maximal torus in $\mathsf{SO}(4)$
- (<u>with</u> justification) the center of $\mathsf{SO}(5)$.

(c) Find the center $Z(\mathsf{G})$ of the group

$$\mathsf{G} = \{1, -1, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\}$$

and hence show that $\mathsf{G}/Z(\mathsf{G})$ has nontrivial center.

[8,8,4]

Question 2. [**16** marks]

Let $A, B, C \in M_n(\mathbb{R})$.

(a) Show that the *exponential series*

$$\mathbf{1} + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

is convergent in $\mathbb{R}^{n^2}$.

(b) Prove that

$$\det\left(e^A\right) = e^{\operatorname{Tr}(A)}.$$

(c) If A is of the form BCB^{-1}, show that $e^A = B\,e^C\,B^{-1}$.

[4,8,4]

Question 3. [**20** marks]

(a) Define the *tangent space* $T_{\mathbf{1}}(\mathsf{G})$ of a matrix (Lie) group G. Hence show that $T_{\mathbf{1}}\mathsf{SO}(n)$ consists of precisely the $n \times n$ real vectors (matrices) X such that $X + X^\top = \mathbf{0}$.

(b) Find the tangent space $T_{\mathbf{1}}(\mathsf{SL}(2,\mathbb{C}))$ and then verify that $T_{\mathbf{1}}(\mathsf{SL}(2,\mathbb{C}))$ is a (real) Lie algebra of dimension three.

(c) Show that the matrix $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ in $\mathsf{SL}(2,\mathbb{C})$ is <u>not</u> equal to e^X for any $X \in T_{\mathbf{1}}(\mathsf{SL}(2,\mathbb{C}))$.
(Hint : Use the *Cayley-Hamilton* theorem to show that $X^2 = -\det(X)\,\mathbf{1}$ whenever $\operatorname{Tr}(X) = 0$. Then show that

$$e^X = \cos(\sqrt{\det(X)}\,\mathbf{1} + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}}\,X$$

and hence derive a contradiction.)

[6,6,8]

Question 4. [**22** marks]

(a) Define the term *ideal* (of a Lie algebra), and then prove that if H is a normal subgroup of a matrix Lie group G, then $T_{\mathbf{1}}\mathsf{H}$ is an ideal of the Lie algebra $T_{\mathbf{1}}\mathsf{G}$.

(b) Show that $\mathfrak{sl}(n, \mathbb{C})$ is an ideal of $\mathfrak{gl}(n, \mathbb{C})$.

(c) Explain what is meant by saying that a Lie algebra is *simple*. Hence prove that $\mathfrak{sl}(n, \mathbb{C})$ is simple.

[6,4,12]

Question 5. [**16** marks]

(a) Define the *matrix logarithm*, and then prove that for any matrix e^X within distance 1 of the identity, $\log\left(e^X\right) = X$.

(b) Prove that for any matrix Lie group G there is a neighborhood $N_\delta(\mathbf{1})$ mapped into $T_{\mathbf{1}}\mathsf{G}$ by $\log$.

[4,12]

Question 6. [**16** marks]

(a) Define the terms *open set, path-connected set* and *compact set* (in $\mathbb{R}^k$). Hence show that $\mathsf{GL}(n, \mathbb{R})$ is open in $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, and that $\mathsf{SO}(n)$ is compact in $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$.

(b) Define the *identity component* of a matrix Lie group, and then prove that if G^0 is the identity component of a matrix Lie group G, then G^0 is a normal subgroup of G.

[8,8]

END OF THE EXAMINATION PAPER