RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2013

MATHEMATICS HONOURS

Examiners : Dr C.C. Remsing	AVAILABLE MARKS : 110
Prof B. Makamba	FULL MARKS : 100
	DURATION : 3 HOURS

GEOMETRY (LIE THEORY)

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**20** marks]

(a) Explain what is meant by saying that a group is *simple*. Hence prove that (the rotation group) $\mathsf{SO}\,(3)$ is simple.

(b) Consider the map

$$\varphi : \mathsf{SU}\,(2) \times \mathsf{SU}\,(2) \to \mathsf{SO}\,(4) \quad (v, w) \mapsto \varphi(v, w)$$

where (the rotation of $\mathbb{H} = \mathbb{R}^4$) $\varphi(v, w)$ is given by $q \mapsto v^{-1}qw$. (Recall that any rotation of $\mathbb{H}$ about the origin is a map of the form $q \mapsto v\,q\,w$, where v and w are unit quaternions.)
Show that

i. φ is a *group homomorphism*.
ii. the kernel of φ has two elements.

(c) Prove that (the rotation group) $\mathsf{SO}\,(4)$ is <u>not</u> simple.

[10,6,4]

Question 2. [**20** marks]

(a) Show that a linear transformation on $\mathbb{C}^n$ preserves the *Hermitian inner product* if and only if its matrix A satisfies the condition $A\bar{A}^\top = \mathbf{1}$.

(b) Define the (special unitary) group $\mathsf{SU}(2)$, and then verify that it is a *group*. Is $\mathsf{SU}(2)$ *path-connected* ? Make a clear statement and then prove it.

(c) Define the *center* $Z(\mathsf{G})$ of a group G, and then determine $Z(\mathsf{SU}(2))$.

[4,10,6]

Question 3. [**18** marks]

(a) Define the *tangent space* $T_{\mathbf{1}}\,\mathsf{G}$ of a matrix (Lie) group G. Hence show that $T_{\mathbf{1}}\mathsf{SU}(n)$ consists of all $n \times n$ complex matrices X such that $X + \bar{X}^\top = \mathbf{0}$ and $\mathrm{Tr}\,(X) = 0$.

(b) Prove that for any square matrix A,

$$\det\left(e^A\right) = e^{\mathrm{Tr}\,(A)}.$$

(c) If A is an $n \times n$ complex matrix such that $A\bar{A}^\top = \mathbf{1}$, show that

$$|\det(A)| = 1.$$

[6,10,2]

Question 4. [**18** marks]

(a) Define the term *ideal* (of a Lie algebra), and then prove that if $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism, then its kernel is an ideal.

(b) Explain what is meant by saying that a Lie algebra is *simple*. Hence prove that $\mathfrak{sl}\,(n, \mathbb{C})$ is simple.

[6,12]

Question 5. [**18** marks]

(a) Define the term *matrix Lie group*, and then prove that if $A'(0)$ is a tangent vector at $\mathbf{1}$ to a matrix Lie group G, then $e^{A'(0)} \in \mathsf{G}$.

(b) Suppose that $\langle A_m \rangle$ is a sequence in a matrix Lie group G such that $A_m \to \mathbf{1}$ as $m \to \infty$ and that $\langle \alpha_m \rangle$ is a sequence of real numbers such that $\frac{A_m - \mathbf{1}}{\alpha_m} \to X$ as $m \to \infty$. Prove that $e^{tX} \in \mathsf{G}$ for all $t \in \mathbb{R}$.

[8,10]

Question 6. [**16** marks]

(a) Show that $\mathsf{GL}\,(n, \mathbb{R})$ is open in $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$.

(b) Let G be a path-connected matrix Lie group. Prove that each element of G has the form

$$e^{X_1} e^{X_2} \cdots e^{X_m}$$

for some $X_1, X_2, \ldots, X_m \in T_{\mathbf{1}}\mathsf{G}$.

[4,12]

END OF THE EXAMINATION PAPER