RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : NOVEMBER 2011

MATHEMATICS & APPLIED MATHEMATICS II

Paper 2

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FULL MARKS : 150
DURATION : 3 HOURS

Section A

MAM 202 - DIFFERENTIAL EQUATIONS

AVAILABLE MARKS : 115
FULL MARKS : 100

NB : All questions may be attempted.

Question 1. [**12** marks]

Consider the following initial-value problem:

$$(y^2+1)y' = 2ty, \quad y(2) = 1.$$

(a) Discuss the existence/uniqueness of a solution to the above IVP. If possible, specify the largest interval on which a solution exists and the largest interval on which it is unique.

(b) Find a solution to the IVP.

[7,5]

Question 2. [**20** marks]

Find a general solution to the following differential equations. If an initial value is specified, solve the IVP. (You may leave the answer in implicit form, but simplify as much as possible.)

(a) $ty' + 5y = 7t^2, \quad y(2) = 5;$

(b) $t^2y' + 2ty = 5y^3;$

(c) $y''' - 3y' - 2 = 0.$

[7,8,5]

Question 3. [**8** marks]

Using the method of Variation of Parameters, find a particular solution y_p to the equation

$$t^2y'' - 4ty' + 6y = t^3$$

if $y_c = At^2 + Bt^3$ is the general solution to the associated homogenous equation.

[8]

Question 4. [**10** marks]

Compute the following Laplace transforms. You may use the attached table of transforms.

(a) $\mathcal{L}\left\{3\sin(2t) + t^3\right\}$;

(b) $\mathcal{L}\left\{t^2e^{3t}\right\}$;

(c) $\mathcal{L}\left\{\frac{\cosh(2t)}{t}\right\}$.

[2,3,5]

Question 5. [**20** marks]

Compute the following inverse Laplace transforms. You may use the attached table of transforms.

(a) $\mathcal{L}^{-1}\left\{\frac{3}{s^3}\right\}$;

(b) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\}$;

(c) $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2}\right\}$;

(d) $\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2+1}{s^2+4}\right)\right\}$;

(e) $\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\}$;

(f) $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\}$.

[2,3,4,4,4,3]

Question 6. [**7** marks]

Solve the following IVP using the Laplace transform method.

$$y'' - 3y' + 2y = e^t, \quad y(0) = y'(0) = 0.$$

[7]

Question 7. [**19** marks]

Consider the following system of differential equations:

$$\begin{aligned} x' &= 3x - 2y \\ y' &= x + y. \end{aligned}$$

(a) Show that the eigenvalues of the above system are equal to $2 \pm i$.

(b) Classify the critical point (0,0) of the system (type and stability).

(c) Find the general solution of the system.

(d) If $x(0) = 4$, $y(0) = 3$, find the solution to the IVP.

[3,4,8,4]

Question 8. [**4** marks]

Write the following differential equation as a first-order linear system of differential equations. Define all the functions you use.

$$y''' - y'' + (t^2 - 1)y' + y = e^t.$$

[4]

Question 9. [**15** marks]

Match the following systems of equations to the correct phase portrait given in the appendix, and classify each system's critical point (type and stability).

$$A: \begin{aligned} x' &= x(x^2 - y^2) \\ y' &= 2x - y + 1 \end{aligned},$$

$$B: \begin{aligned} x' &= x^2 - y^2 + 1 \\ y' &= y^2 - x^2 + 1 \end{aligned},$$

$$C: \begin{aligned} x' &= 2x - y + 1 \\ y' &= 7x - y + 3 \end{aligned}$$

[15]

Section B

MAM 202 - MATHEMATICAL MODELLING

AVAILABLE MARKS : 52
FULL MARKS : 50

NB : All questions may be attempted.

Question 1. [**9** marks]

Biotechnologists use bacteria to produce the hormone insulin, and have determined that *the rate of change of amount of insulin is proportional to the* positive *difference between the number of colonies of bacteria and the amount of insulin produced.* Let $B(t)$ give the number of bacterial colonies present at time t, and $I(t)$ the amount of insulin produced at time t. Assume that *the number of bacterial colonies is always greater than the amount of insulin they produce.*

(a) Set up a differential equation which models the amount of insulin produced at time $I(t)$. (NB: This equation will contain an unknown k).

(b) What kind of equation is this ? Express the general form of such an equation in $x(t)$ in terms of unknown functions $f(t)$ and $g(t)$.

(c) Give the general solution $I(t)$ as a function of time if the initial amount of bacterial colonies at time $t = 0$ is assumed to be 100 and the unknown constant $k = 2$.

(d) If the bacteria produce insulin for an indefinite amount of time, what will the amount of insulin produced tend towards?

[2,2,3,2]

Question 2. [**12** marks]

Consider the interactions between species y, a predator, and species x, its prey:

$$\begin{aligned}\dot{x}(t) &= x(t) - 2y(t)\\ \dot{y}(t) &= 2x(t) + 5y(t).\end{aligned}$$

(a) Establish the system of linear ordinary differential equations which expresses this situation in matrix form $\dot{X} = AX$.

(b) Given that the eigenvalues are $\lambda_1 = \lambda_2 = 3$ and the eigenspace is $\langle(-1, 1)\rangle$, express the general solution for this system.

(c) State the Lotka-Volterra equations for a predator-prey system. In what fundamental way do these differ from the given system ?

(d) Give a steady state of this system.

[2,5,3,2]

Question 3. [**16** marks]

Consider a system of three tanks X, Y and Z of brine solution respectively, where brine from tank X flows through tank Y in order to reach tank Z. At any time n the transition from one tank to another is given according to the table

Movement	Percentage
X to X	40%
X to Y	60%
X to Z	0%
Y to X	20%
Y to Y	30%
Y to Z	50%
Z to X	0%
Z to Y	0%
Z to Z	100%

(a) Set up a system of recurrence relations in matrix form which models this situation.

(b) What are the absorbing states?

(c) Express the matrix S of transition between the non-absorbing states.

In the study of marketing, it is determined that consumers will switch between two brands I and J according to the table:

Movement	Percentage
Brand I to Brand J	20%
Brand I to Brand I	80%
Brand J to Brand I	10%
Brand J to Brand J	90%

(d) Set up a system of recurrence relations in matrix form which models this situation.

(e) Express the equation giving the steady state of the system, the *long-term market share* of each brand. (Hint: The eigenvalues of the matrix A are 1 and 0.7 and the corresponding eigenspace is $\langle(1,2),(-1,1)\rangle$.)

(f) Given that initially 1000 consumers choose brand I and 5000 choose brand J, what are the long term market shares of I and J ?

[2,2,2,2,3,5]

Question 4. [**15** marks]

Consider the systems of equations

$$\begin{aligned}\dot{x}(t) &= x(t) + y(t)\\ \dot{y}(t) &= -x(t) + 3y(t)\end{aligned} \qquad \text{and} \qquad \begin{aligned}\dot{x}(t) &= 3x(t) + y(t)\\ \dot{y}(t) &= x(t) + 3y(t)\end{aligned}$$

(a) Express each system in the form $\dot{X} = AX$.

(b) Determine the eigenvalues and eigenvectors of the matrix A in the first system.

(c) Given that the eigenvalues of the matrix A in second system are $\lambda_1 = 4$, $\lambda_2 = 2$ and its corresponding eigenspace is $\langle(1,1),(1,-1)\rangle$, classify the matrices A expressing both systems in matrix form.

(d) Express the general solutions of both systems.

[4,5,2,4]

Section C

MAM 202 - GEOMETRY

AVAILABLE MARKS : 52
FULL MARKS : 50

NB : All questions may be attempted.

Question 1. [**8** marks]

TRUE or FALSE ?

(a) $(\alpha\,\beta)^{-1} = \alpha^{-1}\beta^{-1}$ for transformations α and β.

(b) $\sigma_A\,\sigma_B\,\sigma_C = \sigma_C\,\sigma_B\,\sigma_A$ for points A, B, C.

(c) An even isometry that fixes two points is the identity.

(d) If $\mathcal{L}$ is any line, then every odd isometry is the product of $\sigma_{\mathcal{L}}$ followed by an even isometry.

[2,2,2,2]

Question 2. [**12** marks]

(a) Define the terms *betweenness, halfturn, glide reflection* and *odd isometry.*

(b) Prove ONLY ONE of the following statements :

- If an isometry fixes three noncollinear points, then the isometry must be the identity.
- If α is an isometry, then $\alpha\,\sigma_P\,\alpha^{-1} = \sigma_{\alpha(P)}$.

[4,8]

Question 3. [**16** marks]

PROVE or DISPROVE :

(a) $\sigma_B \, \sigma_A = \tau^2_{B,A}$ for any points A, B.

(b) The set of all dilatations forms a group.

[8,8]

Question 4. [**16** marks]

Consider the points

$$O = (0,0), \quad A = (1,0), \quad B = \left(1, \frac{1}{\sqrt{3}}\right), \quad C = \left(0, \frac{1}{\sqrt{3}}\right)$$

and the line $\mathcal{L}$ with equation

$$x - \sqrt{3}\, y = 0.$$

(a) Write the equations for each of the following transformations :

i. the halfturn σ_M, where $M = \frac{1}{2}(A + C)$.

ii. the rotation $\rho_{B,30}$.

iii. the reflection $\sigma_{\mathcal{L}}$.

iv. the glide reflection $\gamma = \sigma_{\mathcal{L}} \, \sigma_A$.

(b) Find

i. the axis of the glide reflection γ.

ii. the image of $\mathcal{L}$ under $\rho_{B,30}$.

(c) Show that γ^2 is a translation $\tau = \tau_{A,D}$. Hence find D.

(d) Verify that the quadrilateral $\square\, OABC$ is a rectangle and then determine *with justification* the image of $\square\, OABC$ under α.

[5,4,3,4]

END OF THE EXAMINATION PAPER

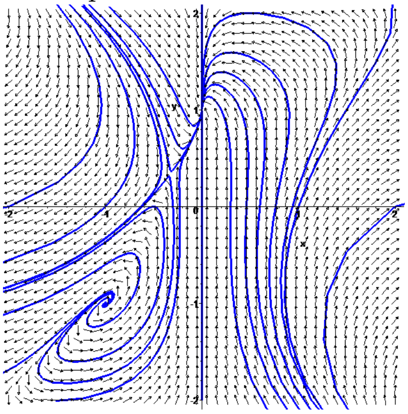
Appendix

Table of Laplace transforms

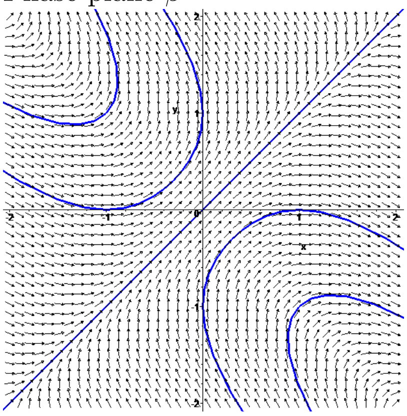
Function	Transform
$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$f'(t)$	$sF(s)-f(0)$
$\int_0^t f(\tau)\mathrm{d}\tau$	$\frac{F(s)}{s}$
$tf(t)$	$F'(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\mathrm{d}\sigma$

Section A: Question 10

Phase plane α



Phase plane β



Phase plane γ