RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2012

MATHEMATICS & APPLIED MATHEMATICS II

Paper 2

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FULL MARKS : 150
DURATION : 3 HOURS

Section A

MAM 201 - LINEAR ALGEBRA

AVAILABLE MARKS : 110
FULL MARKS : 100

NB : All questions may be attempted.

Question 1. [**12** marks]

Use Cramer's Rule to solve the system

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

[12]

Question 2. [**13** marks]

Let V be the set of 2×2 traceless and symmetric matrices, i.e.,

$$V = \left\{ A \in \mathbb{R}^{2\times 2} \,:\, A = A^\top \text{ and } \operatorname{tr}(A) = 0 \right\}.$$

(a) Show that V is closed under the usual operations of scalar multiplication and addition for matrices.

(b) Briefly justify why V is a vector space. (You need not check all the axioms.)

(c) Display a basis for V.

[6,3,4]

Question 3. [**25** marks]

Let V be a vector space.

(a) Define the span of a subset $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ of V.

(b) Prove that the span of S is a subspace of V.

(c) Is $q(x) = x^2 + 4x + 7$ in the span of $\{p_1(x), p_2(x), p_3(x)\}$, where

$$\begin{aligned} p_1(x) &= 2x^3 - x^2 - 2x - 5 \\ p_2(x) &= 3x^3 - x^2 - x - 4 \\ p_3(x) &= -4x^3 + 2x^2 + 4x + 10 \ ? \end{aligned}$$

If so, write $q(x)$ as a linear combination of $p_1(x)$, $p_2(x)$ and $p_3(x)$.

[3,10,12]

Question 4. [**15** marks]

Let $B = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$ be an ordered basis for a vector space V. Prove that any vector $\boldsymbol{v} \in V$ is *uniquely* expressible as a linear combination of the elements in the basis B. Include the following steps.

(a) Define what it means for B to be a basis for V.

(b) Justify that $\boldsymbol{v}$ *can* be written as a linear combination of the elements in the basis B.

(c) Suppose that $\boldsymbol{v}$ can be written as a linear combination of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in two ways. Show that the two expressions for $\boldsymbol{v}$ are identical.

[4,2,9]

Question 5. [**12** marks]

Let P_1 be the vector space of polynomials of degree at most one. Consider the function $T : P_1 \to \mathbb{R}^2$, where $T(p(x)) = (p(0), p(1))$.

(a) Find $T(1 - 2x)$.

(b) Show that T is a linear transformation.

(c) Calculate the kernel and the image of T.

(d) Is T a linear isomorphism?

[2,4,4,2]

Question 6. [**6** marks]

Let $B = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$ be an ordered basis for a vector space V. Find the matrix, with respect to B, of the linear transformation $T : V \to V$ defined by

$$T(\boldsymbol{v}_1) = \boldsymbol{v}_2 + \boldsymbol{v}_3 \qquad T(\boldsymbol{v}_2) = \boldsymbol{v}_3 \qquad T(\boldsymbol{v}_3) = 7\boldsymbol{v}_4 \qquad T(\boldsymbol{v}_4) = \boldsymbol{v}_1 + 3\boldsymbol{v}_2.$$

[6]

Question 7. [**8** marks]

Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2. Show that $\operatorname{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$. (Hint: the characteristic polynomial of A must equal $p_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$.)

[8]

Question 8. [**19** marks]

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}.$$

(a) Find all the eigenvalues and the corresponding eigenspaces of A.

(b) What does it mean to say A is diagonalizable?

(c) Is A diagonalizable? Justify your answer.

[13,3,3]

Section B

MAM 201 - GEOMETRY

AVAILABLE MARKS : 53
FULL MARKS : 50

NB : All questions may be attempted.

Question 1. [**8** marks]

TRUE or FALSE ?

(a) Any translation is a product of two halfturns.

(b) $\rho_{C,r}^{-1} = \rho_{C,-r} = \sigma_C$ for any point C.

(c) If an isometry fixes *three* points, then it *must* be the identity.

(d) An odd isometry that does not fix a point is a glide reflection.

[2,2,2,2]

Question 2. [**12** marks]

(a) Define the terms : *colliniation, even isometry, dilatation* and *glide reflection.*

(b) Prove ONLY ONE of the following two statements :

- The product of two reflections in intersecting lines is a rotation.
- If α is an isometry, then $\alpha\,\tau_{A,B}\,\alpha^{-1} = \tau_{\alpha(A),\alpha(B)}$.

[4,8]

Question 3. [**16** marks]

PROVE or DISPROVE :

(a) The product of *any* two rotations is a rotation.

(b) $\sigma_A \sigma_M = \sigma_M \sigma_B$ if and only if M is the midpoint of A and B.

[8,8]

Question 4. [**17** marks]

Consider the points

$$A = (0, 2), \quad B = (3, 0), \quad C = (2, 1), \quad D = (-1, -1)$$

and the line

$$(\mathcal{L}) \qquad x + y = 1.$$

(a) Write the equations for each of the following transformations :

i. the translation $\tau_{A,B}^{-1}$

ii. the halfturn σ_C

iii. the rotation $\rho_{D,-30}$

iv. the reflection $\sigma_{\mathcal{L}}$

v. the glide reflection $\gamma = \sigma_{\mathcal{L}} \sigma_C$.

(b) Find

i. the image of the point A under σ_C

ii. the preimage of the point B under $\rho_{D,30}^{-1}$

iii. the image of the line $\mathcal{L}$ under $\tau_{B,A}$

iv. the axis of the glide reflection γ.

(c) If $\sigma_{\mathcal{N}} \sigma_{\mathcal{M}}((x, y)) = (x + 6, y - 3)$, find equations for $\mathcal{M}$ and $\mathcal{N}$.

[8,5,4]

Section C

MAM 201 - DISCRETE MATHEMATICS

AVAILABLE MARKS : 53
FULL MARKS : 50

NB : All questions may be answered.

Question 1. [**7** marks]

A playoff between two teams consists of 4 games, the first team that wins three games in a row wins the playoff, if the teams win two games each the playoff is tied. Use tree diagrams, or exhaustive search to find the number of ways in which the playoff can be tied.

Question 2. [**8** marks]

Suppose that $S \cap T = \{a, b, c\}$, $S \cup R = \{a, b, c\}$ and $S \cap R \cap T = \emptyset$ and $T \cup R = \{a, b, c, d\}$.

(a) Find the sets S, R and T.
(b) What is $|\mathcal{P}(S \cap T)|$?
(c) Give a real world example of an empty set.

[6,1,1]

Question 3. [**15** marks]

A photographer at a wedding has to arrange **12** people (including the bride and groom) into **two** rows, each consisting of **4** seats. Give the number of possible seating arrangements for the following.

(a) The bride has to be in the picture.
(b) The groom is not in the picture.
(c) The bride and groom sit in the front row.
(d) The bride and groom sit side by side.
(e) Exactly one of the bride and groom is in the picture.
(f) The bride's dogs Yoko and Ono sit in the front row with the bride and groom. Note that, despite the objections of the bride, the dogs do not count amongst the 12 people.

[2,2,2,3,3,3]

Question 4. [**11** marks]

The South African National Lottery is a game in which a player picks 6 numbers out of a board of 49 numbers ie. from the set $\{1, 2, \ldots, 49\}$. In the draw 7 numbers are chosen without replacement from the set. If the player matches the first 6 numbers drawn the player wins the jackpot prize. The 7th bonus ball number is used for the award of subsequent lesser prizes.

(a) Find the number of possible lottery **entries**.
(b) Find the number of possible lottery **draws** including bonus balls.
(c) Second prize is awarded to a player that has picked 5 correct balls and the bonus ball. Find the number of ways a player can win second place in a given draw (including bonus ball).
(d) Find the number of ways a player can choose three of the winning numbers and the bonus ball in a given draw.
(e) Find the probability that a player who buys 100 tickets loses out on the jackpot prize.

[2,2,2,2,3]

Question 5. [**12** marks]

An urn contains 100 red balls, 100 blue balls and 100 yellow balls.

(a) How many balls must be drawn to give two of the same colour? Which principle justifies this?
(b) How many balls must be drawn to give three of the same colour?
(c) How many ways may 5 balls be drawn from the urn, without replacing the balls as they are drawn?
(d) How many ways may 5 balls be drawn from the urn, with replacment?
(e) Find the probability that a ball chosen at random from the urn is yellow.
(f) If four balls are chosen randomly from the urn without replacement, find the probability that two of the four balls are yellow, and the other two are blue.

[2,2,2,2,2,2]

END OF THE EXAMINATION PAPER