RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 1 : MARCH 2011

AM3.4 (DIFFERENTIAL GEOMETRY)

AVAILABLE MARKS : 52
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1.

(a) Define the following terms : *regular curve, unit-speed curve* and *signed curvature* (of a unit-speed plane curve).

(b) Let $\kappa : (\alpha, \beta) \to \mathbb{R}$ be any smooth function. Show that there is a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ with signed curvature κ.

(c) Let γ be a unit-speed plane curve with nowhere zero curvature. Define the *centre of curvature* $\epsilon(s)$ of γ (at the point $\gamma(s)$) to be

$$\epsilon(s) = \gamma(s) + \frac{1}{\kappa_s(s)}\,\mathbf{n}_s(s).$$

Prove that the circle with centre $\epsilon(s)$ and radius $\frac{1}{|\kappa_s(s)|}$

i. is *tangent* to γ at $\gamma(s)$

ii. has the same *curvature* as γ at $\gamma(s)$.

[4,7,10]

Question 2.

(a) Let γ be a *regular curve* in $\mathbb{R}^3$ with nowhere vanishing curvature. Define carefully the *torsion* of γ. Hence, prove that if the image of γ is contained in a plane, then the torsion τ is zero (at every point of the curve).

(b) Show that the parametrised curve

$$\gamma(t) = \left(\frac{1+t^2}{t},\, t+1,\, \frac{1-t}{t}\right)$$

is *planar*.

[12,6]

Question 3.

(a) Explain what is meant by a *simple closed curve* in $\mathbb{R}^2$ and then carefully define its *length* $\ell(\gamma)$ and the *area* of its interior $\mathcal{A}\,(\mathrm{int}(\gamma))$. Hence, state (but DO NOT PROVE) the *Isoperimetric Inequality.*

(b) By applying the isoperimetric inequality to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a, b > 0)$$

prove that

$$\int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt \geq 2\pi\sqrt{ab}.$$

[7,6]