RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 2 : MAY 2011

AM3.4 (DIFFERENTIAL GEOMETRY)

AVAILABLE MARKS : 53
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1. Define the following terms : *surface patch, atlas* and *smooth surface*. Hence, show that the *unit sphere*

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\right\}$$

is a *smooth surface*. (Give as many details as you possibly can.)

[5,12]

Question 2.

(a) Define the term *level surface* (in $\mathbb{R}^3$). Is it true that any level surface is a smooth surface ? Find a level surface (if any) which is *not* a smooth surface.

(b) Show that the level surface

$$\mathcal{E} = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1\right\}$$

is a *smooth surface*.

(c) Let $f : W \to \mathbb{R}$ be a smooth function. Show that its *graph*

$$\mathcal{G}_f = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, z = f(x, y)\right\}$$

is a *smooth surface* with atlas consisting of a single regular surface path

$$\sigma(u, v) = (u, v, f(u, v)).$$

[4,4,10]

Question 3.

(a) Let $\sigma : U \to \mathbb{R}^3$ be a (regular) surface patch. Define the *first fundamental form* of σ. Hence, calculate the first fundamental form of

$$\sigma(u, v) = (u - v, u + v, u^2 - v^2).$$

What kind of surface is this ?

(b) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two (smooth) surfaces and let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a diffeomorphism. Explain what is meant by saying that f is *conformal* and then state (but DO NOT PROVE) a necessary and sufficient condition for f to be conformal.

(c) Consider the surface patch

$$\sigma(u, v) = (u \cos v, u \sin v, f(u))$$

where $f : W \to \mathbb{R}$ is a smooth function. Find all functions f for which σ is *conformal*.

[6,4,8]