RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 2 : MAY 2012

M3.4 (DIFFERENTIAL GEOMETRY)

AVAILABLE MARKS : 55
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1.

(a) Define the *geodesic curvature* of a regular curve on a surface. Hence compute the geodesic curvature of any circle on a sphere (not necesarily a great circle).

(b) Define the *Weingarten map* $\mathcal{W}_{\mathbf{p}}$ of a surface $\mathcal{S}$ at $\mathbf{p} \in \mathcal{S}$. Let σ be a surface path containing the point $\mathbf{p}$ in its image. Show that the matrix of $\mathcal{W}_{\mathbf{p}}$ with respect to the basis $\{\sigma_u, \sigma_v\}$ of the tangent plane $T_{\mathbf{p}}\mathcal{S}$ is $\mathcal{F}_I^{-1}\mathcal{F}_{II}$. (Here

$$\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \quad \text{and} \quad \mathcal{F}_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

where E, F, G and L, M, N are the coefficients of the first and the second fundamental forms of σ, respectively.)

(c) Let σ be a surface patch with first and second fundamental forms

$$E\,du^2 + 2F\,dudv + G\,dv^2 \quad \text{and} \quad L\,du^2 + 2M\,dudv + N\,dv^2.$$

Show that

$$\sigma_{uu} = \Gamma_{11}^1\sigma_u + \Gamma_{11}^2\sigma_v + L\,\mathbf{N}.$$

Also, compute/derive the coefficient (Christoffel symbol)

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}.$$

[7,8,12]

Question 2.

(a) Define the terms *Gaussian curvature* and *mean curvature* of a surface. Hence compute the Gaussian curvature of (the catenoid)

$$\sigma(u,v) = (\cosh u \cos v, \cosh u \sin v, u).$$

(b) Show that the Gaussian curvature and mean curvature of the surface

$$\mathcal{S} = \big\{(x,y,z) \in \mathbb{R}^3 \,|\, z - f(x,y) = 0\big\}$$

are

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, \quad H = \frac{(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}}{2\sqrt{(1+f_x^2+f_y^2)^3}}.$$

(c) Let $\mathbf{p}$ be a point of a *flat surface* $\mathcal{S}$, and assume that $\mathbf{p}$ is not an umbilic. Prove that there is a surface patch of $\mathcal{S}$ containing $\mathbf{p}$ that is a *ruled surface.*

[8,8,12]