RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 2 : APRIL 2013

MAT314 (DIFFERENTIAL GEOMETRY)

AVAILABLE MARKS : 54
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1.

(a) Define the terms *regular surface patch* and *smooth surface.*

(b) Let U and $\widetilde{U}$ be open subsets of $\mathbb{R}^2$ and let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : \widetilde{U} \to U$ be a bijective smooth map with smooth inverse map. Show that $\widetilde{\sigma} = \sigma \circ \Phi : \widetilde{U} \to \mathbb{R}^3$ is a *regular surface patch.*

(c) Show that, if $f(x,y)$ is a smooth function, its *graph*

$$\Gamma = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, z = f(x,y)\right\}$$

is a *smooth surface* with atlas consisting of a single regular patch

$$\sigma(u,v) = (u,v,f(u,v)).$$

(d) Describe an *atlas* for the surface obtained by rotating the curve

$$x = \cosh z$$

in the xz-plane around the z-axis.

[3,7,4,10]

Question 2.

(a) Explain what is meant by saying that a local diffeomorphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ (between two smooth surfaces) is a *conformal map*. Prove that if there is a function $\lambda : \mathcal{S}_1 \to \mathbb{R}$ such that

$$f^*\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p}} = \lambda(\mathbf{p}) \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p}}$$

for all $\mathbf{p} \in \mathcal{S}_1$ and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}_1$, then (the local diffeomorphism) $f : \mathcal{S}_1 \to \mathcal{S}_2$ is *conformal*. Is the converse true ?

(b) Show that the surface

$$\sigma(u, v) = \left(u - \frac{u^3}{3} + uv^2,\, v - \frac{v^3}{3} + vu^2,\, u^2 - v^2 \right)$$

is *conformally* parametrized.

[15,15]