RHODES UNIVERSITY

DEPARTMENT of MATHEMATICS (Pure & Applied)

CLASS TEST No. 1 : AUGUST 2014

MAT314 (DIFFERENTIAL GEOMETRY)

AVAILABLE MARKS : 54
FULL MARKS : 50
DURATION : 1 HOUR

NB : All questions may be attempted.

Question 1.

(a) Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a parametrized curve (in $\mathbb{R}^3$). Explain what is meant by saying that

   i. γ is *regular*;
   ii. $\widetilde{\gamma}$ is a *reparametrization* of γ;
   iii. $s(\cdot)$ is the *arc-length* of γ (starting at the point $\gamma(t_0)$).

(b) Find the *arc-length* of the parametrized curve

$$\gamma(t) = \left(e^{3t} \cos t,\, e^{3t} \sin t\right), \quad t \in \mathbb{R}$$

starting at the point $(1, 0)$.

(c) Let $\mathbf{p}$ and $\mathbf{q}$ be two points in $\mathbb{R}^3$, and let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a parametrized curve such that $\gamma(a) = \mathbf{p}$ and $\gamma(b) = \mathbf{q}$, where $\alpha < a < b < \beta$. Show that, if $\mathbf{u}$ is any unit vector, then

$$(\mathbf{q} - \mathbf{p}) \bullet \mathbf{u} \leq \int_a^b \|\dot{\gamma}(u)\|\, du.$$

Hence deduce that the length of the part of γ between $\mathbf{p}$ and $\mathbf{q}$ is at least $\|\mathbf{q} - \mathbf{p}\|$.

[4,5,18]

Question 2.

(a) Define the terms *curvature* and *torsion* for a regular (not necessarily unit-speed) space curve. Hence find the curvature and torsion of the parametrized curve

$$\gamma(t) = (2\cos t,\, 2\sin t,\, 3t)\,.$$

(b) Let γ be a regular curve in $\mathbb{R}^2$ and let $\lambda \in \mathbb{R}$. The *parallel curve* γ^λ of γ is defined by

$$\gamma^\lambda(t) = \gamma(t) + \lambda\, \mathbf{n}_s(t).$$

Show that, if $|\lambda\, \kappa_s(t)| < 1$ for all values of t, then γ^λ is a *regular curve* and its *signed curvature* is $\dfrac{\kappa_s}{1 - \lambda\,\kappa_s}$.

[10, 17]