RHODES UNIVERSITY
DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : NOVEMBER 2009

MATHEMATICS III

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AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

AM3.4 - DIFFERENTIAL GEOMETRY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**24** marks]

(a) Let

$$\gamma(t) = (e^{kt}\cos t,\, e^{kt}\sin t), \quad t \in \mathbb{R}.$$

The *arc-length parameter* is defined by

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt} \right\| dt.$$

Find t as a function of s, $t = t(s)$, and then change the parametrisation of the given curve to arc-length parametrisation. What is the *curvature* of γ?

(b) Suppose $\gamma, \tilde{\gamma} : (a,b) \to \mathbb{R}^2$ are two unit-speed curves with the same signed curvatures for all $s \in (a,b)$. Show that that there is a rigid motion M of $\mathbb{R}^2$ such that

$$\tilde{\gamma}(s) = M(\gamma(s))$$

for all $s \in (a,b)$.

(c) Compute the *curvature* κ, the *torsion* τ and the vectors of the Frenet-Serret frame $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ for (the unit-speed curve)

$$\gamma(t) = \left(\frac{4}{5}\cos t,\, 1 - \sin t,\, -\frac{3}{5}\cos t \right).$$

[6,8,10]

Question 2. [**20** marks]

(a) Show that

$$\sigma(u, v) = (\operatorname{sech} u \cdot \cos v,\ \operatorname{sech} u \cdot \sin v,\ \tanh u)$$

is a *regular surface patch* for the unit sphere $\mathbb{S}^2$.

(b) A *loxodrome* is a curve on $\mathbb{S}^2$ that intersects the meridians at a fixed angle, say ϕ. Show that, in the surface patch in (a), a unit-speed loxodrome satisfies

$$\begin{aligned} \dot{u} &= \cos\phi \cdot \cosh u \\ \dot{v} &= \pm \sin\phi \cdot \cosh u. \end{aligned}$$

Deduce that loxodromes correspond under σ to straight lines in the uv-plane.

(c) Explain what is meant by saying that a diffeomorphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ (between two smooth surfaces) is an *isometry*. Then prove that $f : \mathcal{S}_1 \to \mathcal{S}_2$ is an isometry if and only if, for any surface patch σ_1 of $\mathcal{S}_1$, the patches σ_1 and $f \circ \sigma_1$ of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, have the same first fundamental form.

[4,8,8]

Question 3. [**20** marks]

(a) Define the *first* and the *second fundamental forms* of a surface patch. Hence, compute these forms for the elliptic paraboloid

$$\sigma(u, v) = \left(u,\ v,\ u^2 + v^2\right).$$

(b) Prove that if the second fundamental form of a surface patch σ is zero everywhere, then σ is part of a plane.

(c) Define the *normal curvature* of a curve on a surface. Hence, show that the normal curvature of any curve on a sphere of radius r is $\pm \frac{1}{r}$.

[6,8,6]

Question 4. [**24** marks]

(a) Calculate the *principal curvatures* for the torus

$$\sigma(u,v) = ((a + b\cos u)\cos v,\ (a + b\cos u)\sin v,\ b\sin u)\,.$$

(b) Define the terms *gaussian curvature, mean curvature, flat surface*, and *umbilical point*. Hence, prove that if P is a point on a flat surface, so that P is not an umbilical point, then there is a surface patch containing P that is a ruled surface.

(c) Calculate the *gaussian* and *mean curvatures* of the surface

$$\sigma(u,v) = (u + v,\ u - v,\ uv)\,.$$

[6,12,6]

Question 5. [**22** marks]

(a) Define the term *geodesic*, and then write down, without proof, the *geodesic equations*. Explain the connection between geodesics and shortest paths on a arbitrary surface. Make a clear statement but DO NOT prove it.

(b) Consider a surface of revolution

$$\sigma(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u))\,,$$

where we assume that $f > 0$ and that the profile curve $u \mapsto (f(u), 0, g(u))$ is unit-speed. Write down the geodesic equations, and then prove that every meridian $v = v_0$ is a geodesic.

(c) Determine *all* the geodesics on the circular cylinder

$$\mathcal{S} = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\right\}.$$

[6,8,8]

END OF THE EXAMINATION PAPER