RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2010

MATHEMATICS III

Examiners : Dr C.C. Remsing
Prof B. Makamba

AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

AM3.4 - DIFFERENTIAL GEOMETRY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**22** marks]

(a) Explain what is meant by saying that a parametrised curve $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ is *regular*, and then carefully define the *curvature* and the *torsion* of a regular curve (of arbitrary speed). Make clear statements but DO NOT prove them.

(b) Compute the *curvature* and the *torsion* of the circular helix

$$\gamma(t) = (a \cos t,\ a \sin t,\ bt)\,.$$

(c) Let $\kappa, \tau : (\alpha, \beta) \to \mathbb{R}$ be two smooth functions with $\kappa > 0$. Prove that there is a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ whose curvature is κ and whose torsion is τ.

[4,6,12]

Question 2. [**20** marks]

(a) Explain carefully what is meant by saying that $\mathcal{S} \subseteq \mathbb{R}^3$ is a *smooth surface.* Give sufficient conditions for a *level surface*

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$$

to be a smooth surface.

(b) Consider the *hyperbolic paraboloid*

$$\mathcal{H} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 = z\}.$$

Show that $\sigma(u, v) = (u, v, u^2 - v^2)$ is a (regular) parametrisation of the part

$$\mathcal{H}_+ = \{(x, y, z) \in \mathcal{H} \mid z > 0\}$$

of the hyperboloid $\mathcal{H}$. Find another (regular) parametrisation of the same part $\mathcal{H}_+$.

(c) Define the *tangent developable* $\mathcal{T}$ associated with the curve γ. Hence, prove that any tangent developable is isometric to (part of) a plane.

[3,5,12]

Question 3. [**22** marks]

(a) Define the *first* and the *second fundamental forms* of a surface patch. Hence, compute these forms for the surface of revolution

$$\sigma(u, v) = (f(u) \cos v,\ f(u) \sin v,\ g(u)).$$

(You may assume that $f > 0$ and that the profile curve is unit-speed.)

(b) Explain what is meant by saying that a diffeomorfism $f : \mathcal{S}_1 \to \mathcal{S}_2$ (between surfaces) is *conformal.* Hence, prove that if for any surface patch σ_1 on $\mathcal{S}_1$, the first fundamental forms of σ_1 and σ_2 are proportional, then (the diffeomorphism) f is conformal.

(c) Show that the (Mercator's) parametrisation of the sphere

$$\sigma(u, v) = (\mathrm{sech\,u} \cdot \mathrm{cos\,v},\ \mathrm{sech\,u} \cdot \mathrm{sin\,v},\ \mathrm{tanh\,u})$$

is conformal.

[6,10,6]

Question 4. [**24** marks]

(a) Define the terms *principal curvatures, gaussian curvature*, and *mean curvature*. Hence, calculate the principal curvatures and the gaussian curvature for the helicoid

$$\sigma(u, v) = (v \cos u,\ v \sin u,\ \lambda\, u).$$

(b) A curve γ on a smooth surface $\mathcal{S}$ is called *asymptotic* if its normal curvature is everywhere zero. Hence, show that a curve γ with positive curvature is asymptotic if and only if its binormal $\mathbf{b}$ is parallel to the unit normal of $\mathcal{S}$ at all points of γ.

(c) Show that the *asymptotic curves* on the surface

$$\sigma(u, v) = (u \cos v,\ u \sin v,\ \ln u)$$

are given by

$$\ln u = \pm\,(v + c), \quad c \in \mathbb{R}.$$

[8,6,10]

Question 5. [**22** marks]

(a) Define the term *geodesic*, and then write down, without proof, the *geodesic equations.* Explain the connection between geodesics and shortest paths on a arbitrary surface. Make a clear statement but DO NOT prove it.

(b) Describe *four different* geodesics on the hyperboloid of one sheet

$$\mathcal{H} = \left\{ (x, y, z) \in \mathbb{R}^3 \ \mid \ x^2 + y^2 - z^2 = 1 \right\}$$

passing through the point $(1, 0, 0)$. (Hint : Find two lines and two normal sections.)

(c) Determine *all* the geodesics on the circular cone

$$\mathcal{S} = \left\{ (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = z^2 \right\}.$$

[6,8,8]

END OF THE EXAMINATION PAPER