RHODES UNIVERSITY
DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2011

MATHEMATICS III

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AVAILABLE MARKS : 106
FULL MARKS : 100
DURATION : 3 HOURS

AM3.4 - DIFFERENTIAL GEOMETRY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**26** marks]

(a) Explain what is meant by a *reparametrisation* of a parametrised curve $\gamma : (\alpha, \beta) \to \mathbb{R}^3$. Is it true that any parametrised curve has a unit-speed reparametrisation ? Make a clear statement (but DO NOT prove it). Hence, find a unit-speed reparametrisation of the circle

$$\gamma(t) = (a + R\cos t, b + R\sin t\ , 0).$$

(b) Compute the *curvature* of the parametrised (plane) curve

$$\gamma(t) = (t, \cosh t)\,.$$

(CAUTION: The curve is <u>not</u> unit-speed.)

(c) Let $\gamma, \widetilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ be two unit-speed curves with the same curvature $\kappa(s) > 0$ and the same torsion $\tau(s)$ for all $s \in (\alpha, \beta)$. Prove that there is a rigid motion M of $\mathbb{R}^3$ such that

$$\widetilde{\gamma}(s) = M(\gamma(s)), \quad s \in (\alpha, \beta).$$

[7,5,14]

Question 2. [**20** marks]

(a) Define the *first fundamental form* of a surface patch. Hence, compute this form for

$$\sigma(u, v) = (\cosh u, \sinh u, v)\,.$$

(b) Explain what is meant by saying that a diffeomorphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ (between surfaces) is an *isometry*. Hence, prove that if (the diffeomorphism) f is an isometry, then for any surface patch σ_1 on $\mathcal{S}_1$, the patches σ_1 and $f \circ \sigma_1$ of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, have the same first fundamental form.

(c) Prove that any (generalised) cylinder is isometric to (part of) a plane.

[6,8,6]

Question 3. [**20** marks]

(a) Define the *normal curvature* and the *geodesic curvature* of a unit-speed curve. Hence, show that the normal curvature of any curve on a sphere of radius r is $\pm\frac{1}{r}\cdot$

(b) Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve on a surface path σ. Show that its *normal curvature* is given by

$$\kappa_n = L\,\dot{u}^2 + 2M\,\dot{u}\dot{v} + N\,\dot{v}^2$$

where $L\,du^2 + 2M\,du\,dv + N\,dv^2$ is the second fundamental form of σ.

(c) State (but DO NOT PROVE) *Euler's Theorem* (expressing the normal curvature of a curve on a surface patch in terms of the principal curvatures). Hence, derive the fact that the principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point.

[8,6,6]

Question 4. [**20** marks]

(a) Define the terms *gaussian curvature* and *mean curvature*. Hence, calculate the gaussian curvature for the surface

$$\mathcal{S} = \left\{ (x, y, z) \in \mathbb{R}^3 \,|\, z = f(x, y) \right\}$$

where $f : W \to \mathbb{R}$ is a smooth function.

(b) Show that the gaussian curvature of a ruled surface

$$\sigma(u, v) = \gamma(u) + v\,\delta(u)$$

is negative or zero.

(c) Prove that for any compact surface $\mathcal{S}$, there is a point $P \in \mathcal{S}$ at which the gaussian curvature K is positive.

[6,6,8]

Question 5. [**20** marks]

(a) Define the term *geodesic*, and then write down, without proof, the *geodesic equations*. Hence, prove that on the surface of revolution

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

every meridian is a geodesic.

(b) A surface of revolution has the property that *every* parallel is a geodesic. What kind of surface is it ? Justify your answer.

(c) Determine *all* the geodesics on the circular cylinder

$$\mathcal{S} = \left\{ (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1 \right\}.$$

[6,6,8]

END OF THE EXAMINATION PAPER