RHODES UNIVERSITY
DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2012

MATHEMATICS III

Examiners : Dr C.C. Remsing	AVAILABLE MARKS : 110
Prof B. Makamba	FULL MARKS : 100
	DURATION : 3 HOURS

M3.4 - DIFFERENTIAL GEOMETRY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**24** marks]

(a) Explain what is meant by a *regular curve* (in $\mathbb{R}^3$) and then define the *curvature* of a regular curve. Hence compute the curvature of the (regular) curve

$$\gamma(t) = (\cos t, \, \sin t, \, t) \, .$$

(b) Define the *torsion* of a regular curve. Hence prove that, given a regular curve γ with nowhere vanishing curvature, the image of γ is contained in a plane <u>if and only if</u> its torsion τ is zero at every point of the curve.

(c) Let (a_{ij}) be a skew-symmetric 3×3 matrix. Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ be smooth functions (of a parameter s) satisfying the differential equations

$$\dot{\mathbf{v}}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + a_{i3}\mathbf{v}_3, \quad i = 1, 2, 3$$

and suppose that, for some particular value s_0, the vectors $\mathbf{v}_1(s_0)$, $\mathbf{v}_2(s_0)$ and $\mathbf{v}_3(s_0)$ are orthogonal. Prove that the vectors $\mathbf{v}_1(s)$, $\mathbf{v}_2(s)$ and $\mathbf{v}_3(s)$ are orthogonal <u>for all</u> values of s.

[5,7,12]

Question 2. [**18** marks]

(a) Explain what is meant by saying that $\mathcal{S} \subseteq \mathbb{R}^3$ is a *smooth surface.* Hence show that the circular cylinder

$$\mathcal{S} = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\right\}$$

is a smooth surface.

(b) Define the *first fundamental form* of a surface $\mathcal{S}$ (at $\mathbf{p} \in \mathcal{S}$). Hence compute this form for

$$\sigma(u, v) = \left(u - v, u + v, u^2 + v^2\right).$$

What kind of surface is this ?

(c) Explain what is meant by saying that a smooth map $f : \mathcal{S}_1 \to \mathcal{S}_2$ (between surfaces) is a *local isometry.* Hence prove that $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a local isometry <u>if and only if</u> the symmetric bilinear forms $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ and $f^*\langle \cdot, \cdot \rangle$ on $T_{\mathbf{p}}\mathcal{S}_1$ are equal for all $\mathbf{p} \in \mathcal{S}_1$. (Here $f^*\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p}} = \langle D_{\mathbf{p}} f(\mathbf{v}), D_{\mathbf{p}} f(\mathbf{w}) \rangle_{f(\mathbf{p})}$ for $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}_1$.)

[7,4,7]

Question 3. [**22** marks]

(a) Define the *second fundamental form* of a surface patch. Hence compute this form for

$$\sigma(u, v) = (u, v, u^2 + v^2).$$

(b) Define the *normal curvature* and the *geodesic curvature* of a regular, but <u>not necesarily</u> unit-speed, curve on a surface. Hence prove that (with the usual notation) the normal and geodesic curvatures of such a regular curve γ are

$$\kappa_n = \frac{\langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle}{\langle \dot{\gamma}, \dot{\gamma} \rangle} \quad \text{and} \quad \kappa_g = \frac{\ddot{\gamma} \bullet (\mathbf{N} \times \dot{\gamma})}{\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle^3}}.$$

(c) Let γ be a (smooth) curve on a surface $\mathcal{S}$ and let $\mathbf{p}, \mathbf{q} \in \mathcal{S}$. Define carefully the *parallel transport* $\Pi_\gamma^{\mathbf{pq}} : T_{\mathbf{p}}\mathcal{S} \to T_{\mathbf{q}}\mathcal{S}$, and then show that $\Pi_\gamma^{\mathbf{pq}}$ is a linear isometry.

[5,7,10]

Question 4. [**24** marks]

(a) Define the terms *principal curvatures, principal vectors* and *Gaussian curvature* of a surface. Hence calculate the principal curvatures, principal vectors and the Gaussian curvature for (the cylinder)

$$\sigma(u, v) = (\cos v, \sin v, u).$$

(b) Let $\mathcal{S}$ be a (connected) surface of which every point is an umbilic. Prove that $\mathcal{S}$ is an open subset of a plane or a sphere.

(c) A curve γ on a surface $\mathcal{S}$ is called a *line of curvature* if the tangent vector of γ is a principal vector of $\mathcal{S}$ at all points of γ. Show that $\gamma(t) = \sigma(u(t), v(t))$ (on a surface patch σ) is a line of curvature <u>if and only if</u> (in the usual notation)

$$(EM - FL)\,\dot{u}^2 + (EN - GL)\,\dot{u}\dot{v} + (FN - GM)\,\dot{v}^2 = 0.$$

[6,10,8]

Question 5. [**22** marks]

(a) Define the term *geodesic* and then prove that a curve on a surface is a geodesic <u>if and only if</u> its geodesic curvature is zero everywhere. Hence justify the claim that all great circles on a sphere are geodesics.

(b) A (regular) curve γ with nowhere vanishing curvature on a surface $\mathcal{S}$ is called a *pre-geodesic* on $\mathcal{S}$ if some reparametrization of γ is a geodesic on $\mathcal{S}$. Show that a curve γ is a pre-geodesic <u>if and only if</u>

$$\ddot{\gamma} \bullet (\mathbf{N} \times \dot{\gamma}) = 0$$

everywhere on γ.

(c) Prove that any point of a surface of constant Gaussian curvature is contained in a patch that is isometric to an open subset of a plane, a sphere or a pseudosphere.

[5,5,12]

END OF THE EXAMINATION PAPER