RHODES UNIVERSITY
DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : JUNE 2013

MATHEMATICS III

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AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

MAT 314 - DIFFERENTIAL GEOMETRY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**22** marks]

(a) Define the *curvature* of a regular curve, and then compute the curvature of

$$\gamma(t) = (t,\, \cosh t,\, 0).$$

(b) Let $\gamma, \widetilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ be two unit-speed curves with the same *signed curvature* for all $s \in (\alpha, \beta)$. Prove that there is a rigid motion M of $\mathbb{R}^2$ such that

$$\widetilde{\gamma}(s) = M(\gamma(s))$$

for all $s \in (\alpha, \beta)$.

(c) Compute the *curvature* κ, the *torsion* τ and the vectors of the Serret-Frenet frame $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ for the (unit-speed) curve

$$\gamma(t) = \left(\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}} \right).$$

[4,8,10]

Question 2. [**22** marks]

(a) Define the *first fundamental form* of a surface, and then compute this form for (the elliptic paraboloid)

$$\sigma(u,v) = \big(u, v, u^2 + v^2\big)\,.$$

(b) Prove that a smooth map (between smooth surfaces) $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a *local isometry* if and only if the symmetric bilinear forms $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ and $f^*\langle \cdot, \cdot \rangle_{\mathbf{p}}$ on (the tangent plane) $T_{\mathbf{p}}\mathcal{S}_1$ are equal for all $\mathbf{p} \in \mathcal{S}_1$.

(c) Let $\Phi : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^2$. Write $\Phi(u,v) = (f(u,v),\, g(u,v))$, where f and g are smooth functions on the uv-plane. Prove that Φ is *conformal* if and only if

either $(f_u = g_v$ and $f_v = -g_u)$ or $(f_u = -g_v$ and $f_v = g_u)$.

[4,6,12]

Question 3. [**24** marks]

(a) Define the *second fundamental form* of a surface patch, and then compute this form for (the surface of revolution)

$$\sigma(u,v) = (f(u)\cos v,\; f(u)\sin v,\; g(u))$$

where $u \mapsto (f(u), 0, g(u))$ is a unit-speed curve.

(b) Define the *Gaussian curvature* of a surface, and then give a formula (for the Gaussian curvature) in terms of the coefficients of the first and the second fundamental forms of an admissible patch. Hence compute the Gaussian curvature of the surface patch in (a).

(c) Prove that if $\mathcal{S}$ is a *compact* surface, then there exists a point of $\mathcal{S}$ at which its Gaussian curvature K is > 0.

[6,10,8]

Question 4. [**20** marks]

(a) Define the *principal curvatures* and the *principal vectors* of a surface. Hence calculate the principal curvatures and the principal vectors for (the cylinder)

$$\sigma(u,v) = (\cos v, \sin v, u).$$

(b) Let γ be a curve on the (oriented) surface $\mathcal{S}$, and let κ_1 and κ_2 be the principal curvatures of $\mathcal{S}$, with non-zero principal vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. Prove that the *normal curvature* of γ is given by

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

where θ is the oriented angle $\widehat{\mathbf{t}_1\dot{\gamma}}$.

(c) Compute the *mean curvature* of the surface

$$z = f(x,y)$$

where f is a smooth function.

[6,8,6]

Question 5. [**22** marks]

(a) Define the term *geodesic* and then prove that a curve on a surface is a geodesic <u>if and only if</u> its geodesic curvature is zero everywhere.

(b) Prove that a curve γ on a surface $\mathcal{S}$ is a *geodesic* <u>if and only if</u>, for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a surface patch σ, the so-called geodesic equations are satisfied

$$\begin{aligned} \frac{d}{dt}(E\dot{u} + F\dot{v}) &= \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2) \\ \frac{d}{dt}(F\dot{u} + G\dot{v}) &= \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2) \end{aligned}$$

where $E\,du^2 + 2F\,dudv + G\,dv^2$ is the first fundamental form of σ.

(c) Use the geodesic equations in (b) to determine the geodesics of a *sphere*.

[4,8,10]

END OF THE EXAMINATION PAPER