RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS (Pure & Applied)

EXAMINATION : NOVEMBER 2014

MATHEMATICS III

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AVAILABLE MARKS : 110
FULL MARKS : 100
DURATION : 3 HOURS

MAT 314 - DIFFERENTIAL GEOMETRY

NB : All questions may be attempted. All steps must be clearly motivated. Marks will not be awarded if this is not done.

Question 1. [**26** marks]

(a) Explain what is meant by a *unit-speed reparametrization* of a parametrized curve. Prove that if a parametrized curve has a unit-speed reparametrization, then it is a regular curve. Is the converse also true ?

(b) Find a unit-speed reparametrization of the curve

$$\gamma(t) = \left(e^t \cos t,\, e^t \sin t\right).$$

(c) Let $\kappa, \tau : (\alpha, \beta) \to \mathbb{R}$ be two smooth functions with $\kappa > 0$. Prove that there is a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ whose curvature is κ and whose torsion is τ.

[8,6,12]

Question 2. [**22** marks]

(a) Write down *parametrizations* of each of the following quadrics:

- $\dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} = \dfrac{z^2}{r^2}$
- $\dfrac{x^2}{p^2} - \dfrac{y^2}{q^2} = 1.$

(b) Define the *first fundamental form* of a surface, and then compute this form for

$$\sigma(u, v) = (u \cos v, u \sin v, f(u)) .$$

(c) Explain what is meant by saying that a smooth map (between surfaces) $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a *local isometry.* Hence prove that $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a local isometry <u>if and only if</u> the symmetric bilinear forms $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ and $f^* \langle \cdot, \cdot \rangle_{\mathbf{p}}$ on (the tangent plane) $T_{\mathbf{p}}\mathcal{S}_1$ are equal for all $\mathbf{p} \in \mathcal{S}_1$. (Here $f^* \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p}} = \langle D_{\mathbf{p}} f(\mathbf{v}), D_{\mathbf{p}} f(\mathbf{w}) \rangle_{f(\mathbf{p})}$ for $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}_1$.)

[4,6,12]

Question 3. [**24** marks]

(a) Define the *second fundamental form* of a surface patch, and then compute this form for (the surface of revolution)

$$\sigma(u, v) = (f(u) \cos v,\ f(u) \sin v,\ g(u))$$

where $u \mapsto (f(u), 0, g(u))$ is a unit-speed curve.

(b) Let γ be a curve on a surface $\mathcal{S}$ and let $\mathbf{p}, \mathbf{q} \in \mathcal{S}$. Define carefully the *parallel transport* $\Pi_\gamma^{\mathbf{pq}} : T_{\mathbf{p}}\mathcal{S} \to T_{\mathbf{q}}\mathcal{S}$ from $\mathbf{p}$ to $\mathbf{q}$ along γ, and then prove that $\Pi_\gamma^{\mathbf{pq}}$ is a linear isometry.

(c) Let $E\, du^2 + 2F\, dudv + G\, dv^2$ be the first fundamental form of a surface patch σ of a surface $\mathcal{S}$. Show that, if $\mathbf{p}$ is a point in the image of σ and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$, then

$$\langle \mathbf{v}, \mathbf{w} \rangle = E\, du(\mathbf{v}) du(\mathbf{w}) + F\, (du(\mathbf{v}) dv(\mathbf{w}) + du(\mathbf{w}) dv(\mathbf{v})) + G\, dv(\mathbf{v}) dv(\mathbf{w}).$$

[6,12,6]

Question 4. [**24** marks]

(a) Define the *principal curvatures* and the *principal vectors* of a surface. Hence calculate the principal curvatures and the principal vectors for

$$\sigma(u,v) = (\cos v, \sin v, u).$$

(b) Let γ be a curve on an oriented surface $\mathcal{S}$, and let κ_1 and κ_2 be the principal curvatures of $\mathcal{S}$, with (non-zero) principal vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. Prove that the normal curvature of γ is given by

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

where θ is the oriented angle between $\mathbf{t}_1$ and $\dot{\gamma}$.

(c) Calculate the *Gaussian* and *mean curvatures* of the surface

$$\sigma(u,v) = (u+v, u-v, uv)$$

at the point $(2,0,1)$.

[6,8,10]

Question 5. [**14** marks]

(a) Define the term *geodesic* and then prove that a curve on a surface is a geodesic <u>if and only if</u> its geodesic curvature is zero everywhere.

(b) Prove that a curve γ on a surface $\mathcal{S}$ is a *geodesic* <u>if and only if</u>, for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a surface patch σ, the so-called geodesic equations are satisfied

$$\begin{aligned} \frac{d}{dt}(E\dot{u} + F\dot{v}) &= \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2) \\ \frac{d}{dt}(F\dot{u} + G\dot{v}) &= \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2) \end{aligned}$$

where $E\,du^2 + 2F\,dudv + G\,dv^2$ is the first fundamental form of σ.

[4,10]

END OF THE EXAMINATION PAPER