Key Results

Differential Geometry (MAT314)

C.C. Remsing
Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown 6140

(Curves)

- (proposition 1.1.6; p.5): *If the tangent vector of a parametrized curve is constant, the image of the curve is (part of) a straight line.* [**1**]

- (proposition 1.2.4; p.11): *Let* $\mathbf{n}(t)$ *be a unit vector that is a smooth function of a parameter* t*. Then, the dot product* $\dot{\mathbf{n}}(t) \bullet \mathbf{n}(t) = 0$ *for all* t*, i.e.,* $\dot{\mathbf{n}}(t)$ *is zero or perpendicular to* $\mathbf{n}(t)$ *for all* t*. In particular, if* γ *is a unit-speed curve, then* $\ddot{\gamma}$ *is zero or perpendicular to* $\dot{\gamma}$. [**2**]

- (proposition 1.3.4; p.14): *Any reparametrization of a regular curve is regular.* [**3**]

- (proposition 1.3.5; p.14): *If* $\gamma(t)$ *is a regular curve, its arc-length* $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\|\, du$*, starting at any point of* γ*, is a smooth function of* t. (NO PROOF)

- (proposition 1.3.6; p.15): *A parametrized curve has a unit-speed reparametrization <u>if and only if</u> it is regular.* [**4**]

- (corollary 1.3.7; p.16): *Let* γ *be a regular curve and let* $\widetilde{\gamma}$ *be a unit-speed reparametrization of* γ *:* $\widetilde{\gamma}(u(t)) = \gamma(t)$ *for all* t*, where* u *is a smooth function of* t*. Then, if* s *is the arc-length of* γ *(starting at any point), we have* $u = \pm s + c$*, where* c *is a constant. Conversely, if* u *is given by* $u = \pm s + c$ *for some value of* c *and with either sign, then* $\widetilde{\gamma}$ *is a unit-speed reparametrization of* γ. [**5**]

- (theorem 1.5.1; p.23): *Let* $f(x, y)$ *be a smooth function of two variables. Asume that, at every point of the level curve* $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \,|\, f(x, y) = 0\}$*,* $\partial f/\partial x$ *and* $\partial f/\partial y$ *are not both zero. If* $\mathbf{p}$ *is a point of* $\mathcal{C}$*, with coordinates* (x_0, y_0)*, say, there is a regular parametrized curve* $\gamma(t)$*, defined on an open interval containing* 0*, such that* γ *passes through* $\mathbf{p}$ *when* $t = 0$ *and* $\gamma(t)$ *is contained in* $\mathcal{C}$ *for all* t*.* (NO PROOF)

- (theorem 1.5.2; p.26): *Let* γ *be a regular parametrized plane curve, and let* $\gamma(t_0) = (x_0, y_0)$ *be a point in the image of* γ*. Then, there is a smooth real-valued function* $f(x, y)$*, defined for* x *and* y *in open intervals containing* x_0 *and* y_0*, respectively, and satisfying the conditions in Theorem 1.5.1, such that* $\gamma(t)$ *is contained in the level curve* $f(x, y) = 0$ *for all values of* t *in some open interval containing* t_0*.* (NO PROOF)

- (proposition 2.1.2; p.31): *Let* $\gamma(t)$ *be a regular curve in* $\mathbb{R}^3$*. Then, its curvature is* $\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$. [**6**]

- (proposition 2.2.1; p.36): *Let* $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ *be a unit-speed curve, let* $s_0 \in (\alpha, \beta)$ *and let* φ_0 *be such that* $\dot{\gamma}(s_0) = (\cos\varphi_0, \sin\varphi_0)$*. Then there exists a unique function* $\varphi : (\alpha, \beta) \to \mathbb{R}$ *such that* $\varphi(s_0) = \varphi_0$ *and that* $\dot{\varphi}(t) = (\cos\varphi(s), \sin\varphi(s))$ *holds for all* $s \in (\alpha, \beta)$*.* (NO PROOF)

- (proposition 2.2.3; p.38) *Let* $\gamma(s)$ *be a unit-speed plane curve, and let* $\varphi(s)$ *be the turning angle for* γ*. Then* $\kappa_s = \frac{d\varphi}{ds}$. [**7**]

- (theorem 2.2.6; p.39); *Let* $\kappa : (\alpha, \beta) \to \mathbb{R}$ *be any smooth function. Then, there is a unit-speed curve* $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ *whose signed curvature is* κ*. Further, if* $\widetilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ *is any other unit-speed curve whose signed curvature is* κ*, there is a direct isometry* M *of* $\mathbb{R}^2$ *such that* $\widetilde{\gamma}(s) = M(\gamma(s))$ *for all* $s \in (\alpha, \beta)$*.* [**8**]

- (proposition 2.3.1; p.48): *Let* $\gamma(t)$ *be a regular curve in* $\mathbb{R}^3$ *with nowhere vanishing curvature. Then, its torsion is given by* $\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \bullet \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$.

  [**9**]

- (proposition 2.3.3; p.49): *Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then, the image of γ is contained in a plane* <u>*if and only if*</u> *τ is zero at every point of the curve.* [**10**]

- (theorem 2.3.4; p.50): *Let γ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then, $\dot{\mathbf{t}} = \kappa\,\mathbf{n}$, $\dot{\mathbf{n}} = -\kappa\,\mathbf{t} + \tau\,\mathbf{b}$ and $\dot{\mathbf{b}} = -\tau\,\mathbf{n}$.*

  [**11**]

- (proposition 2.3.5; p.51): *Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then, γ is a parametrization of (part of) a circle.* [**12**]

- (theorem 2.3.6; p.52): *Let $\gamma(s)$ and $\widetilde{\gamma}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same curvature $\kappa(s) > 0$ and the same torsion $\tau(s)$ for all s. Then, there is a direct isometry M of $\mathbb{R}^3$ such that $\widetilde{\gamma}(s) = M(\gamma(s))$ for all s. Further, if κ and τ are smooth functions with $\kappa > 0$ everywhere, there is a unit-speed curve in $\mathbb{R}^3$ whose curvature is κ and whose torsion is τ.* [**13**]

(Surfaces)

- (proposition 4.2.6; p.78): *The transition maps of a smooth surface are smooth.* (NO PROOF)

- (proposition 4.2.7; p.78): *Let U and $\widetilde{U}$ be open subsets of $\mathbb{R}^3$ and let $\sigma : U \to \mathbb{R}^3$ be a regular surface pach. Let $\Phi : \widetilde{U} \to U$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : U \to \widetilde{U}$. Then, $\widetilde{\sigma} = \sigma \circ \Phi : \widetilde{U} \to \mathbb{R}^3$ is a regular surface patch.* [**14**]

- (proposition 4.3.1; p.83): *Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a diffeomorphism. If σ_1 is an allowable surface patch on $\mathcal{S}_1$, then $f \circ \sigma_1$ is an allowable surface patch on $\mathcal{S}_2$.* [**15**]

- (proposition 4.4.2; p.85): *Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface $\mathcal{S}$ containing a point $\mathbf{p}$, and let (u, v) be coordinates in U. The tangent space to $\mathcal{S}$ at $\mathbf{p}$ is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v (the derivatives are evaluated at the point $(u_0, v_0) \in U$ such that $\sigma(u_0, v_0) = \mathbf{p}$).* [**16**]

- (proposition 4.4.4; p.87): *If* $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ *is a smooth map between surfaces and* $\mathbf{p} \in \mathcal{S}$*, the derivative* $D_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\widetilde{\mathcal{S}}$ *is a linear map.* [**17**]

- (proposition 4.4.5; p.88): *(i) If* $\mathcal{S}$ *is a surface and* $\mathbf{p} \in \mathcal{S}$*, the derivative at* $\mathbf{p}$ *of the identity map* $\mathcal{S} \to \mathcal{S}$ *is the identity map* $T_{\mathbf{p}}\mathcal{S} \to T_{\mathbf{p}}\mathcal{S}$*.*
  (ii) If $\mathcal{S}_1, \mathcal{S}_2$ *and* $\mathcal{S}_3$ *are surfaces and* $f_1 : \mathcal{S}_1 \to \mathcal{S}_2$ *and* $f_2 : \mathcal{S}_2 \to \mathcal{S}_3$ *are smooth maps, then for all* $\mathbf{p} \in \mathcal{S}_1$*,* $D_{\mathbf{p}}(f_2 \circ f_1) = D_{f_1(\mathbf{p})}f_2 \circ D_{\mathbf{p}}f_1$*.*
  (iii) If $f : \mathcal{S}_1 \to \mathcal{S}_2$ *is a diffeomorphism, then for all* $\mathbf{p} \in \mathcal{S}_1$ *the linear map* $D_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S}_1 \to T_{f(\mathbf{p})}\mathcal{S}_2$ *is invertible.* [**18**]

- (proposition 4.4.6; p.88): *Let* $\mathcal{S}$ *and* $\widetilde{\mathcal{S}}$ *be surfaces and let* $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ *be a smooth map. Then,* f *is a local diffeomorphism <u>if and only if</u>, for all* $\mathbf{p} \in \mathcal{S}$*, the linear map* $D_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\widetilde{\mathcal{S}}$ *is invertible.* (NO PROOF)

- (proposition 4.5.2; p.90): *Let* $\mathcal{S}$ *be an orientable surface equipped with an atlas* $\mathcal{A}$ *as in Definition 4.5.1. Then, there is a smooth choice of unit normal at any point of* $\mathcal{S}$*: take the standard unit normal of any surface patch in* $\mathcal{A}$. [**19**]

- (theorem 5.1.1; p.95): *Let* $\mathcal{S}$ *be a subset of* $\mathbb{R}^3$ *with the following property: for each point* $\mathbf{p} \in \mathcal{S}$*, there is an open subset* W *of* $\mathbb{R}^3$ *containing* $\mathbf{p}$ *and a smooth function* $f : W \to \mathbb{R}$ *such that*
  (i) $\mathcal{S} \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$*;*
  (ii) The gradient $\nabla f = (f_x, f_y, f_z)$ *of* f *does not vanish at* $\mathbf{p}$*.*
  Then, $\mathcal{S}$ *is a smooth surface.* (NO PROOF)

- (theorem 5.2.2; p.97): *By applying a direct isometry of* $\mathbb{R}^3$*, every non-empty quadric* $\mathbf{v}^\top A\mathbf{v} + \mathbf{b}^\top\mathbf{v} + c = 0$ *in which the coefficients are not all zero can be transformed into one whose Cartesian equation is one of the following:*
  (i) Ellipsoid: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$.
  (ii) Hyperboloid of one sheet: $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$.
  (iii) Hyperboloid of two sheets: $\frac{x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$.
  (iv) Elliptic paraboloid: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$.
  (v) Hyperbolic paraboloid: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$.
  (vi) Quadric cone: $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$.

(vii) Elliptic cylinder: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$.
(viii) Hyperbolic cylinder: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$.
(ix) Parabolic cylinder: $\frac{x^2}{p^2} = y$.
(x) Plane: $x = 0$.
(xi) Two parallel planes: $x^2 = p^2$.
(xii) Two intersecting planes: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 0$.
(xiii) Straight line: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 0$.
(xiv) Single point: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 0$.
In each case, $p, q,$ *and* r *are non-zero constants.* (NO PROOF)

- (theorem 5.4.4; p.119: *For any integer* $g \geq 0$, T_g *has an atlas making it a smooth surface. Moreover, every compact surface is diffeomorphic to one of the* T_g. (NO PROOF)

- (corollary 5.4.5; p.119): *Every compact surface is orientable.* [**20**]

- (theorem 6.2.2; p.127): *A smooth map* $f : \mathcal{S}_1 \to \mathcal{S}_2$ *is a local isometry if and only if the symmetric bilinear forms* $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ *and* $f^*\langle \cdot, \cdot \rangle_{\mathbf{p}}$ *on* $T_{\mathbf{p}}\mathcal{S}_1$ *are equal for all* $\mathbf{p} \in \mathcal{S}_1$. [**21**]

- (corollary 6.2.3; p.128): *A local diffeomorphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ *is a local isometry if and only if, for any patch* σ_1 *of* $\mathcal{S}_1$*, the patches* σ_1 *and* $f \circ \sigma_1$ *of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, respectively, have the same first fundamental form.* [**22**]

- (proposition 6.2.5; p.130): *Any tangent developable is locally isometric to a plane.* [**23**]

- (theorem 6.3.3; p.134): *A local diffeomorphism* $f ; \mathcal{S}_1 \to \mathcal{S}_2$ *is conformal if and only if there is a function* $\lambda : \mathcal{S}_1 \to \mathbb{R}$ *such that* $f^*\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p}} = \lambda(\mathbf{p})\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p}}$ *for all* $\mathbf{p} \in \mathcal{S}_1$ *and* $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}_1$.*(It is not hard to see that the function* λ*, if it exists, is necessarily smooth).* [**24**]

- (corollary 6.3.4; p.136): *A local diffeomorphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ *is conformal if and only if, for any surface patch* σ *of* $\mathcal{S}_1$*, the first fundamental forms of the patches* σ *of* $\mathcal{S}_1$ *and* $f \circ \sigma$ *of* $\mathcal{S}_2$ *are proportional.* [**25**]

- (theorem 6.3.6; p.138): *Every surface has an atlas consisting of conformal surface patches.* (NO PROOF)

(Curvature)

- (proposition 7.2.2; p.164): *Let* $\mathbf{p}$ *be a point of a surface* $\mathcal{S}$*, let* $\sigma(u,v)$ *be a surface patch of* $\mathcal{S}$ *with* $\mathbf{p}$ *in its image, and let* $L\,du^2+2M\,dudv+N\,dv^2$ *be the second fundamental form of* σ*. Then, for any* $\mathbf{v},\mathbf{w}\in T_{\mathbf{p}}\mathcal{S}$*,*

  $$\langle\langle \mathbf{v},\mathbf{w}\rangle\rangle = L\,du(\mathbf{v})du(\mathbf{w})+M\,(du(\mathbf{v})dv(\mathbf{w})+du(\mathbf{w})dv(\mathbf{v}))+N\,dv(\mathbf{v})dv(\mathbf{w}).$$

  [**26**]

- (lemma 7.2.3; p.164): *Let* $\sigma(u,v)$ *be a surface patch with standard unit normal* $\mathbf{N}(u,v)$*. Then,*

  $$\mathbf{N}_u\bullet\sigma_u=-L,\quad \mathbf{N}_u\bullet\sigma_v=\mathbf{N}_v\bullet\sigma_u=-M,\quad \mathbf{N}_v\bullet\sigma_v=-N$$

  where $L=\sigma_{uu}\bullet\mathbf{N}$, $M=\sigma_{uv}\bullet\mathbf{N}$ *and* $N=\sigma_{vv}\bullet\mathbf{N}$. [**27**]

- (corollary 7.2.4; p.165): *The second fundamental form is a symmetric bilinear form. Equivalently, the Weingarten map is self-adjoint.* [**28**]

- (proposition 7.3.2; p.166): *With the above notation, we have*

  $$\kappa_n=\ddot{\gamma}\bullet\mathbf{N},\quad \kappa_g=\ddot{\gamma}\bullet(\mathbf{N}\times\dot{\gamma}),\quad \kappa^2=\kappa_n^2+\kappa_g^2$$
  $$\kappa_n=\kappa\,\cos\psi,\quad \kappa_g=\pm\kappa\,\sin\psi$$

  where κ *is the curvature of* γ *and* ψ *is the angle between* $\mathbf{N}$ *and the principal normal* $\mathbf{n}$ *of* γ. [**29**]

- (proposition 7.3.3; p.167): *If* γ *is a unit-speed curve on an oriented surface* $\mathcal{S}$*, its normal curvature is given by* $\kappa_n=\langle\langle\dot{\gamma},\dot{\gamma}\rangle\rangle$*. If* σ *is a surface patch of* $\mathcal{S}$ *and* $\gamma(t)=\sigma(u(t),v(t))$ *is a curve in* σ*,* $\kappa_n=L\,\dot{u}^2+2M\,\dot{u}\dot{v}+N\,\dot{v}^2$. [**30**]

- (corollary 7.3.5; p.169): *The curvature* κ*, the normal curvature* κ_n *and geodesic curvature* κ_g *of a normal section of a surface are related by* $\kappa_n=\pm\,\kappa$*,* $\kappa_g=0$. [**31**]

- (propositionn 7.4.3; p.171): *A tangent vector field* $\mathbf{v}$ *is parallel along a curve* γ *on a surface* $\mathcal{S}$ *if and only if* $\dot{\mathbf{v}}$ *is perpendicular to the tangent plane of* $\mathcal{S}$ *at all points of* γ. [**32**]

- (proposition 7.4.4 (**Gauss Equations**); p.172): *Let* $\sigma(u,v)$ *be a surface patch with first and second fundamental forms* $E\,du^2+2F\,dudv+G\,dv^2$ *and* $L\,du^2+2M\,dudv+n\,dv^2$. *Then,*

$$\begin{aligned}\sigma_{uu} &= \Gamma^1_{11}\sigma_u+\Gamma^2_{11}\sigma_v+L\,\mathbf{N}\\ \sigma_{uv} &= \Gamma^1_{12}\sigma_u+\Gamma^2_{12}\sigma_v+M\,\mathbf{N}\\ \sigma_{vv} &= \Gamma^1_{22}\sigma_u+\Gamma^2_{22}\sigma_v+N\,\mathbf{N}\end{aligned}$$

where

$$\Gamma^1_{11}=\frac{GE_u-2FF_u+FE_v}{2(EG-F^2)}\qquad \Gamma^2_{11}=\frac{2EF_u-EE_v-FE_u}{2(EG-F^2)}$$
$$\Gamma^1_{12}=\frac{GE_v-FG_u}{2(EG-F^2)}\qquad \Gamma^2_{12}=\frac{EG_u-FE_v}{2(EG-F^2)}$$
$$\Gamma^1_{22}=\frac{2GF_v-GG_u-FG_v}{2(EG-F^2)}\qquad \Gamma^2_{22}=\frac{EG_v-2FF_v+FG_u}{2(E-F^2)}.$$

(The six Γ *coefficients in these formulas are called Christoffel symbols.)*

[**33**]

- (proposition 7.4.5; p.173): *Let* $\gamma(t)=\sigma(u(t),v(t))$ *be a curve on a surface patch* σ, *and let* $\mathbf{v}(t)=\alpha(t)\,\sigma_u+\beta(t)\,\sigma_v$ *be a tangent vector field along* γ, *where* α *and* β *ar smooth functions of* t. *Then,* $\mathbf{v}$ *is parallel along* γ *if and only if the following equations are satisfied:*

$$\begin{aligned}\dot\alpha+(\Gamma^1_{11}\dot u+\Gamma^1_{12}\dot v)\,\alpha+(\Gamma^1_{12}\dot u+\Gamma^1_{22}\dot v)\,\beta &= 0\\ \dot\beta+(\Gamma^2_{11}\dot u+\Gamma^2_{12}\dot v)\,\alpha+(\Gamma^2_{12}\dot u+\Gamma^2_{22}\dot v)\,\beta &= 0.\end{aligned}$$

[**34**]

- (corollary 7.4.6; p.174): *Let* γ *be a curve on a surface* $\mathcal{S}$ *and let* $\mathbf{v}_0$ *be a tangent vector of* $\mathcal{S}$ *at the point* $\gamma(t_0)$. *Then, there is exactly one tangent vector field* $\mathbf{v}$ *that is parallel along* γ *and is such that* $\mathbf{v}(t_0)=\mathbf{v}_0$. [**35**]

- (proposition 7.4.9; p.175): *With the notation in Definition 7.4.8,*
  (i) $\Pi^{\mathbf{pq}}_\gamma$ *is a linear map.*
  (ii) $\Pi^{\mathbf{pq}}_\gamma$ *is an isometry, i.e., it preserves lengths and angles.* [**36**]

- (proposition 8.1.2; p.180): *Let* σ *be a surface patch of an oriented surface* $\mathcal{S}$*. Then, with the above notation, the matrix of* $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ *with respect to the basis* $\{\sigma_u, \sigma_v\}$ *of* $T_{\mathbf{p}}\mathcal{S}$ *is* $\mathcal{F}_I^{-1}\mathcal{F}_{II}$. [**37**]

- (corollary 8.1.3; p.181): *We have*

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} \quad \textit{and} \quad K = \frac{LN - M^2}{EG - F^2}.$$

[**38**]

- (proposition 8.2.1; p.187): *Let* $\mathbf{p}$ *be a point of a surface* $\mathcal{S}$*. There are scalars* κ_1, κ_2 *and a basis* $\{\mathbf{t}_1, \mathbf{t}_2\}$ *of the tangent plane* $T_{\mathbf{p}}\mathcal{S}$ *such that* $\mathcal{W}(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1$, $\mathcal{W}(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2$*. Moreover, if* $\kappa_1 \neq \kappa_2$*, then* $\langle \mathbf{t}_1, \mathbf{t}_2 \rangle = 0$.

[**39**]

- (corollary 8.2.2; p.187): *If* $\mathbf{p}$ *is a point of a surface* $\mathcal{S}$*, there is an orthonormall basis of the tangent plane* $T_{\mathbf{p}}\mathcal{S}$ *consisting of principal vectors.* [**40**]

- (proposition 8.2.3; p.188): *If* κ_1 *and* κ_2 *are the principal curvature of a surface, the mean and Gaussian curvatures are given by* $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ *and* $K = \kappa_1\kappa_2$. [**41**]

- (proposition 8.2.4 (**Euler's Theorem**); p.188): *Let* γ *be a curve on an oriented surface* $\mathcal{S}$*, and let* κ_1 *and* κ_2 *be the principal curvatures of* σ*, with non-zero principal vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$*. Then, the normal curvature of* γ *is* $\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$*, where* θ *is the oriented angle* $\widehat{\mathbf{t}_1\dot{\gamma}}$. [**42**]

- (corollary 8.2.5; p.189): *The principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Moreover, the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.* [**43**]

- (proposition 8.2.6; p.190): *The principal curvatures are the roots of the equation*

$$\begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa f & N - \kappa G \end{vmatrix} = 0$$

and the principal vectors corresponding to the principal curvature κ *are the tangent vectors* $\mathbf{t} = \xi\,\sigma_u + \eta\,\sigma_v$ *such that*

$$\begin{pmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

[**44**]

- (proposition 8.2.9; p.191): *Let* $\mathcal{S}$ *be a (connected) surface of which every point is an umbilic. Then,* $\mathcal{S}$ *is an open subset of a plane or a sphere.* [**45**]

- (proposition 8.4.1; p.201): *Let* $\mathbf{p}$ *be a point of a surface* $\mathcal{S}$*, and suppose that* $\mathbf{p}$ *is not an umbilic. Then, there is a surface patch* $\sigma(u,v)$ *of* $\mathcal{S}$ *containing* $\mathbf{p}$ *whose first and second fundamental forms are* $E\,du^2 + G\,dv^2$ *and* $L\,du^2 + N\,dv^2$*, respectively, for some smooth functions* E, G, L *and* N. (NO PROOF)

- (proposition 8.4.2; p.201): *Let* $\mathbf{p}$ *be a point of a flat surface* $\mathcal{S}$*, and assume that* $\mathbf{p}$ *is not an umbilic. Then, there is a patch of* $\mathcal{S}$ *containing* $\mathbf{p}$ *that is a ruled surface.* [**46**]

- (proposition 8.5.2; p.207): *Let* κ_1 *and* κ_2 *be the principal curvatures of an oriented surface* $\mathcal{S}$*, let* $\lambda \in \mathbb{R}$ *and let* $\mathcal{S}^\lambda$ *be the corresponding parallel surface of* $\mathcal{S}$*. Assume that neither* κ_1 *nor* κ_2 *is equal to* $1/\lambda$ *at any point of* $\mathcal{S}$*. Then,*
  (i) $\mathcal{S}^\lambda$ *is (smooth) oriented surface, the unit normal of* $\mathcal{S}^\lambda$ *at* $\mathbf{p}+\lambda\,\mathbf{N_p}$ *being equal to* $\epsilon\,\mathbf{N_p}$*, where* ϵ *is the sign of* $(1-\lambda\kappa_1)(1-\lambda\kappa_2)$*.*
  (ii) The principal curvatures of $\mathcal{S}^\lambda$ *are* $\epsilon\,\kappa_1/(1-\lambda\kappa_1)$ *and* $\epsilon\,\kappa_2/(1-\lambda\kappa_2)$*, and the corresponding principal vectors are the same as those of* $\mathcal{S}$ *for the principal curvatures* κ_1 *and* κ_2*, respectively.*
  (iii) The Gaussian and mean curvatures of $\mathcal{S}^\lambda$ *are*
  $$\frac{K}{1-2\lambda H+\lambda^2 K} \quad \textit{and} \quad \frac{\epsilon\,(H-\lambda K)}{1-2\lambda H+\lambda^2 K}$$
  respectively, where K *and* H *and the Gaussian and mean curvatures of* $\mathcal{S}$. [**47**]

- (corollary 8.5.3; p.209): *If* $\mathcal{S}$ *has constant Gaussian curvature* $1/R^2$*, the parallel surfaces* $\mathcal{S}^{\pm R}$ *have constant mean curvature* $1/2R$*. Conversely, if* $\mathcal{S}$ *has constant mean curvature* $\mp\epsilon/(2R)$*, the parallel surface* $\mathcal{S}^R$ *has constant Gaussian curvature* $1/R^2$. [**48**]

- (proposition 8.6.1; p.212): *If $\mathcal{S}$ is a compact surface, there is a point of $\mathcal{S}$ at which its Gaussian curvature K is > 0 .* [**49**]

(Geodesics)

- (proposition 9.1.2; p.216): *Any geodesic has constant speed.* [**50**]
- (proposition 9.1.3; p.216): *A unit-speed curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere.* [**51**]
- (proposition 9.1.4; p.217): *Any (part of a) straight line on a surface is a geodesic.* [**52**]
- (proposition 9.1.5; p.216): *All straight lines in the plane are geodesics, as are the rulings of any ruled surface, such as those of a (generalized) cylinder or a (generalized) cone, or the straight lines on a hyperboloid of one sheet.* [**53**]
- (proposition 9.1.6; p.218): *Any normal section of a surface is a geodesic.*

[**54**]

- (theorem 9.2.1; p.220): *A curve γ on a surface $\mathcal{S}$ is a geodesic if and only if, for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a surface patch σ of $\mathcal{S}$, the following two equations are satisfied:*

$$\begin{aligned} \frac{d}{dt}\left(E\,\dot{u} + F\,\dot{v}\right) &= \frac{1}{2}\left(E_u\,\dot{u}^2 + 2F_u\,\dot{u}\dot{v} + G_u\,\dot{v}^2\right) \\ \frac{d}{dt}\left(F\,\dot{u} + G\,\dot{v}\right) &= \frac{1}{2}\left(E_v\,\dot{u}^2 + 2F_v\,\dot{u}\dot{v} + G_v\,\dot{v}^2\right) \end{aligned}$$

where $E\,du^2 + 2F\,dudv + G\,dv^2$ is the first fundamental form of σ.

[**55**]

- (proposition 9.2.3; p.223): *A curve γ on a surface $\mathcal{S}$ is a geodesic if and only if, for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a surface patch σ of $\mathcal{S}$, the following two equations are satisfied:*

$$\begin{aligned} \ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 &= 0 \\ \ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 &= 0. \end{aligned}$$

[**56**]

- (proposition 9.2.4; p.223): *Let* $\mathbf{p}$ *be a point of a surface* $\mathcal{S}$*, and let* $\mathbf{t}$ *be a unit tangent vector to* $\mathcal{S}$ *at* $\mathbf{p}$*. Then, there exists a unique unit-speed geodesic* γ *on* $\mathcal{S}$ *which passes through* $\mathbf{p}$ *and has tangent vector* $\mathbf{t}$ *there.* [**57**]

- (corollary 9.2.7; p.224): *Any local isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.* [**58**]

- (proposition 9.3.1; p.227): *On the surface of revolution* $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$,
  (i) Every meridian is a geodesic.
  (ii A parallel $u = u_0$ *is a geodesic* <u>*if and only if*</u> $df/du = 0$ *when* $u = u_0$*, i.e.,* u_0 *is a stationary point of* f*.* [**59**]

- (proposition 9.3.2 (**Clairaut's Theorem**); p.228): *Let* γ *be a unit-speed curve on a surface of revolution* $\mathcal{S}$*, let* $\rho : \mathcal{S} \to \mathbb{R}$ *be the distance of a point of* $\mathcal{S}$ *from the axis of rotation, and let* ψ *be the angle between* $\dot{\gamma}$ *and the meridians of* $\mathcal{S}$*. If* γ *is a geodesic, then* $\rho \sin\psi$ *is constant along* γ*. Conversely, if* $\rho\sin\psi$ *is constant allong* γ*, and if no part of* γ *is part of some parallel of* $\mathcal{S}$*, then* γ *is a geodesic.* [**60**]

- (theorem 9.4.1; p.237): *With the above notation, the unit-speed curve* γ *is a geodesic* <u>*if and only if*</u> $\frac{d}{d\tau}\mathcal{L}(\tau) = 0$ *when* $\tau = 0$ *for all families of curves* γ^τ *with* $\gamma^0 = \gamma$*.* (NO PROOF)