Key Words

Differential Geometry (MAT314)

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(Curves)

- (p.2): A **level curve** (in $\mathbb{R}^2$) is a set of points $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$.
- (definition 1.1.1; p.2): A **parametrized curve** (in $\mathbb{R}^n$) is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.
- (definition 1.1.5; p.4): If γ is a parametrized curve, the first derivative $\dot{\gamma}(t)$ is called the **tangent vector** of γ at the point $\gamma(t)$.
- (definition 1.2.1; p.10): The **arc-length** of a (parametrized) curve γ starting at the point $\gamma(t_0)$ is the function $s(t)$ given by $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| \, du$.
- (definition 1.2.3; p.11): If $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is a parametrized curve, its **speed** at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$, and γ is said to be a **unit-speed curve** if $\dot{\gamma}(t)$ is a unit vector for all $t \in (\alpha, \beta)$.
- (definition 1.3.1; p.13): A parametrized curve $\widetilde{\gamma} : (\widetilde{\alpha}, \widetilde{\beta}) \to \mathbb{R}^n$ is a **reparametrization** of a parametrized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi : (\widetilde{\alpha}, \widetilde{\beta}) \to (\alpha, \beta)$ (the **reparametrization map**) such that the inverse map $\phi^{-1} : (\alpha, \beta) \to (\widetilde{\alpha}, \widetilde{\beta})$ is also smooth and $\widetilde{\phi}(\widetilde{t}) = \gamma(\phi(\widetilde{t}))$ for all $\widetilde{t} \in (\widetilde{\alpha}, \widetilde{\beta})$.
- (definition 1.3.3; p.13): A point $\gamma(t)$ of a parametrized curve γ is called a **regular point** if $\dot{\gamma}(t) \neq \mathbf{0}$; otherwise $\gamma(t)$ is a **singular point** of γ. A curve is **regular** if all its points are regular.

- (definition 2.1.1; p.30): If γ is a unit-speed curve with parameter t, its **curvature** $\kappa(t)$ at the point $\gamma(t)$ is defined to be $\|\ddot{\gamma}(t)\|$.
- (p.34): There are two unit vectors perpendicular to (the tangent vector) $\mathbf{t}$; we make a choice by defining $\mathbf{n}_s$, the **signed unit normal** of γ, to be the unit vector obtained by rotating $\mathbf{t}$ anticlockwise by $\pi/2$.
- (p.34): There is a scalar κ_s such that $\ddot{\gamma} = \kappa_s \mathbf{n}_s$; κ_s is called the **signed curvature** of γ (it can be positive, negative or zero).
- (definition 2.2.2; p.36): The (unique) smooth function $\varphi : (\alpha, \beta) \to \mathbb{R}$ such that $\dot{\gamma}(s) = (\cos\varphi(s), \sin\varphi(s))$ for all $s \in (\alpha, \beta)$ and $\varphi(s_0) = \varphi_0$ is called the **turning angle** of γ determined by the condition $\varphi(s_0) = \varphi_0$.
- (p.45): We define the **principal normal** of (the unit-speed curve) γ at the point $\gamma(s)$ to be the vector $\mathbf{n}(s) = \frac{1}{\kappa(s)}\,\dot{\mathbf{t}}(s)$.
- (p.45): The vector $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$ is called the **binormal vector** of (the unit-speed curve) γ at the point $\gamma(s)$.
- (p.46): $\dot{\mathbf{b}}$ is parallel to $\mathbf{n}$, so $\dot{\mathbf{b}} = -\tau\,\mathbf{n}$ for some scalar τ, which is called the **torsion** of γ. (The torsion is only defined if the curvature is non-zero.)

(Surfaces)

- (p.67): A subset U of $\mathbb{R}^n$ is called **open** if, whenever $\mathbf{a}$ is a point in U, there is a positive number ϵ such that every point $\mathbf{u} \in \mathbb{R}^n$ within a distance ϵ of $\mathbf{a}$ is also in U:

$$\mathbf{a} \in U \quad \text{and} \quad \|\mathbf{u} - \mathbf{a}\| < \epsilon \quad \Longrightarrow \quad \mathbf{u} \in U.$$

- (p.68): The (open) set $\mathcal{D}_r(\mathbf{a}) = \{\mathbf{u} \in \mathbb{R}^n \mid \|\mathbf{u} - \mathbf{a}\| < r\}$ is called the **open ball** with centre $\mathbf{a}$ and radius $r > 0$. (If $n = 1$, an open ball is called an **open interval**; if $n = 2$ it is called an **open disc**.)
- (p.68): A map $f : X \subseteq \mathbb{R}^m \to Y \subseteq \mathbb{R}^n$ is said to be **continuous** at $\mathbf{a} \in X$ if, given any number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\mathbf{u} \in X \quad \text{and} \quad \|\mathbf{u} - \mathbf{a}\| < \delta \quad \Longrightarrow \quad \|f(\mathbf{u}) - f(\mathbf{a})\| < \epsilon.$$

Then f is said to be **continuous** if it is continuous at every point of X.

- (p.68): If (the map) $f : X \to Y$ is continuous and bijective, and its inverse map $f^{-1} : Y \to X$ is also continuous, then f is called a **homeomorphism** (and X and Y are said to be **homeomorphic**).

- (definition 4.1.1; p.68): A subset $\mathcal{S}$ of $\mathbb{R}^3$ is a **surface** if, for every point $\mathbf{p} \in \mathcal{S}$, there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing $\mathbf{p}$ such that $\mathcal{S} \cap W$ is homeomorphic to U. A subset of a surface $\mathcal{S}$ of the form $\mathcal{S} \cap W$, where W is an open subset of $\mathbb{R}^3$, is called an **open subset** of $\mathcal{S}$. A homeomorphism $\sigma : U \to \mathcal{S} \cap W$ (as in this definition) is called a **surface patch** (or **parametrization**) of the open subset $\mathcal{S} \cap W$ of $\mathcal{S}$. A collection of such surface patches whose images cover the whole of $\mathcal{S}$ is called an **atlas** of $\mathcal{S}$.

- (example 4.1.3; p.69): The **unit cylinder** is the set (smooth surface) $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$.

- (p.71): $\sigma(\theta, \varphi) = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ \sin\theta)$ is the **latitude-longitude parametrization** of the unit sphere $\mathbb{S}^2$.

- (p.72): The composite homeomorphism

$$\sigma^{-1} \circ \widetilde{\sigma} : \widetilde{\sigma}^{-1}(\mathcal{S} \cap W \cap \widetilde{W}) \to \sigma^{-1}(\mathcal{S} \cap W \cap \widetilde{W})$$

  is called the **transition map** from (the surface patch) $\sigma : U \to \mathcal{S} \cap W$ to (the surface patch) $\widetilde{\sigma} : \widetilde{U} \to \mathcal{S} \cap \widetilde{W}$.

- (definition 4.2.1; p.75): A surface patch $\sigma : U \to \mathbb{R}^3$ is called **regular** if it is smooth and the vectors σ_u and σ_v are linearly independent at all points $(u, v) \in U$. (Equivalently, σ should be smooth and the vector product $\sigma_u \times \sigma_v$ should be non-zero at every point of U.)

- (definition 4.2.2; p.75): If $\mathcal{S}$ is a surface, an **allowable surface patch** for $\mathcal{S}$ is a regular surface patch $\sigma : U \to \mathbb{R}^3$ such that σ is a homeomorphism from U to an open subset of $\mathcal{S}$. A **smooth surface** is a surface $\mathcal{S}$ such that, for any point $\mathbf{p} \in \mathcal{S}$, there is an allowable surface patch σ as above such that $\mathbf{p} \in \sigma(U)$. A collection $\mathcal{A}$ of allowable surface patches for a surface $\mathcal{S}$ such that every point of $\mathcal{S}$ is in the image of at least one patch in $\mathcal{A}$ is called an **atlas** for the smooth surface $\mathcal{S}$.

- (p.83): We say that the map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is **smooth** if (for every surface patches $\sigma_1 : U_1 \to \mathbb{R}^3$ and $\sigma_2 : U_2 \to \mathbb{R}^3$ of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively) the map $\sigma_2^{-1} \circ f \circ \sigma_1 : U_1 \to U_2$ is smooth.

- (p.83): A smooth map $f : \mathcal{S}_1 \to \mathcal{S}_2$, which is bijective and whose inverse map $f^{-1} : \mathcal{S}_2 \to \mathcal{S}_1$ is smooth, is called a **diffeomorphism**. $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be **diffeomorphic** if there is a diffeomorphism between them.

- (p.83): A smooth map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is called a **local diffeomorphism** if, for any point $\mathbf{p} \in \mathcal{S}_1$, there is an open subset $\mathcal{O}$ of $\mathcal{S}_1$ such that $f(\mathcal{O})$ is an open subset of $\mathcal{S}_2$ and $f|_{\mathcal{O}} : \mathcal{O} \to f(\mathcal{O})$ is a diffeomorphism (note that open subsets of surfaces are surfaces).

- (definition 4.4.1; p.85): A **tangent vector** to a surface $\mathcal{S}$ at a point $\mathbf{p} \in \mathcal{S}$ is the tangent vector at $\mathbf{p}$ of a curve in $\mathcal{S}$ passing through $\mathbf{p}$. The **tangent space** $T_{\mathbf{p}}\mathcal{S}$ of $\mathcal{S}$ at $\mathbf{p}$ is the set of all tangent vectors to $\mathcal{S}$ at $\mathbf{p}$.

- (definition 4.4.3; p.87): The **derivative** $D_{\mathbf{p}}f$ of (the smooth map) $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ at the point $\mathbf{p} \in \mathcal{S}$ is the map $D_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\widetilde{\mathcal{S}}$ such that $D_{\mathbf{p}}f(\mathbf{w}) = \widetilde{\mathbf{w}}$ for any tangent vector $\mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$.

- (p.89): The **standard unit normal** of the surface patch σ at $\mathbf{p}$ is the vector $\mathbf{N}_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$.

- (definition 4.5.1; p.90): A surface $\mathcal{S}$ is **orientable** if there exists an atlas $\mathcal{A}$ for $\mathcal{S}$ with the property that, if Φ is the transition map between any two surface patches in $\mathcal{A}$, then $\det\left(J(\Phi)\right) > 0$ where Φ is defined.

- (p.95): A **level surface** (in $\mathbb{R}^3$) is a set of the form $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$.

- (definition 5.2.1; p.97): A **quadric** is the subset of $\mathbb{R}^3$ defined by an equation of the form $\mathbf{v}^\top A\mathbf{v} + \mathbf{b}^\top \mathbf{v} + c = 0$, where $\mathbf{v} = (x, y, z)$, A is a constant symmetric 3×3 matrix, $\mathbf{b} \in \mathbb{R}^3$ is a constant vector, and c is constant scalar.

- (example 5.3.1; p.104): A **ruled surface** is a surface that is a union of straight line, called the **rulings** of the surface.

- (example 5.3.1; p.105): A **generalized cylinder** is a special case of a ruled surface in which the rulings are all parallel to each other.

- (example 5.3.1; p.106): A **generalized cone** (with vertex $\mathbf{v}$) is a (second) special case of a ruled surface in which the rulings all pass through a certain fixd point $\mathbf{v}$.

- (example 5.3.2; p.107): A **surface of revolution** is the surface obtained by rotating a plane curve, called the **profile curve**, around a straight line in the plane.

- (p.109): A subset X of $\mathbb{R}^3$ is called **compact** if it is **closed** (i.e., the set of points in $\mathbb{R}^3$ that are not in X is open) and **bounded** (i.e., X is contained in some open ball).

- (definition 6.1.1; p.122): Let $\mathbf{p}$ be a point of a surface $\mathcal{S}$. The **first fundamental form** of $\mathcal{S}$ at $\mathbf{p}$ associates to tangent vectors $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$ the scalar $\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{p},\mathcal{S}} = \mathbf{v} \bullet \mathbf{w}$.

- (definition 6.2.1; p.125): If $\mathcal{S}_1$ and $\mathcal{S}_2$ are surfaces, a smooth map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is called a **local isometry** if it takes any curve in $\mathcal{S}_1$ to a curve of the same length in $\mathcal{S}_2$. If a local isometry $f : \mathcal{S}_1 \to \mathcal{S}_2$ exists, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are **locally isometric**.

- (p.132): Suppose that two curves γ and $\widetilde{\gamma}$ on a surface $\mathcal{S}$ intersect at a point $\mathbf{p}$. The **angle** θ of intersetion of γ and $\widetilde{\gamma}$ at $\mathbf{p}$ is defined to be the angle between the tangent vectors $\dot{\gamma}$ and $\dot{\widetilde{\gamma}}$ (evaluated at $t = t_0$ and $t = \widetilde{t}_0$, respectively).

- (definition 6.3.2; p.133): If $\mathcal{S}_1$ and $\mathcal{S}_2$ are surfaces, a **conformal map** $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a local diffeomorphism such that, if γ_1 and $\widetilde{\gamma}_1$ are any two curves on $\mathcal{S}_1$ that intersect at a point $\mathbf{p} \in \mathcal{S}_1$, and if γ_2 and $\widetilde{\gamma}_2$ are their images under f, the angle of intersection of γ_1 and $\widetilde{\gamma}_1$ at $\mathbf{p}$ is equal to the angle of intersection of γ_2 and $\widetilde{\gamma}_2$ at $f(\mathbf{p})$. (In short, f is conformal if and only if it preserves angles.)

(Curvature)

- (p.160): One calls the expression $L\,du^2 + 2M\,dudv + N\,dv^2$, where

$$L = \sigma_{uu} \bullet \mathbf{N}, \quad M = \sigma_{uv} \bullet \mathbf{N}, \quad N = \sigma_{vv} \bullet \mathbf{N}$$

  the **second fundamental form** of the surface patch σ.

- (p.162): The **Gauss map** $\mathcal{G} = \mathcal{G}_{\mathcal{S}}$ is the map from (the oriented surface) $\mathcal{S}$ to the unit sphere $\mathbb{S}^2$ that assigns to any point $\mathbf{p} \in \mathcal{S}$ the point $\mathbf{N}_{\mathbf{p}} \in \mathbb{S}^2$, where $\mathbf{N}_{\mathbf{p}}$ is the unit normal of $\mathcal{S}$ at $\mathbf{p}$.

- (definition 7.2.1; p.163): Let $\mathbf{p}$ be a point of a surface $\mathcal{S}$. The **Weingarten map** $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ of $\mathcal{S}$ at $\mathbf{p}$ is defined by $\mathcal{W}_{\mathbf{p},\mathcal{S}} = -D_{\mathbf{p}}\mathcal{G}$.

- (definition 7.2.1; p.163): The **second fundamental form** of $\mathcal{S}$ at $\mathbf{p}$ is the bilinear form on $T_{\mathbf{p}}\mathcal{S}$ given by

$$\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_{\mathbf{p},\mathcal{S}} = \langle \mathcal{W}_{\mathbf{p},\mathcal{S}}(\mathbf{v}), \mathbf{w} \rangle_{\mathbf{p},\mathcal{S}}$$

- (definition 7.3.1); p.166): The scalars κ_n and κ_g in equation $\ddot{\gamma} = \kappa_n\,\mathbf{N} + \kappa_g\,\mathbf{N} \times \dot{\gamma}$ are called the **normal curvature** and the **geodesic curvature** of γ, respectively.

- (p.169): A **normal section** of a surface $\mathcal{S}$ is a curve γ which is the intersection of $\mathcal{S}$ with a plane Π that is perpendicular to the tangent plane of the surface at every point of γ.

- (definition 7.4.1; p.171): Let γ be a curve on a surface $\mathcal{S}$ and let $\mathbf{v}$ be a tangent vector field along γ. The **covariant derivative** of $\mathbf{v}$ along γ is the orthogonal projection $\nabla_\gamma \mathbf{v}$ of $d\mathbf{v}/dt$ onto the tangent plane $T_{\gamma(t)}\mathcal{S}$ at the point $\gamma(t)$.

- (definition 7.4.2; p.171): (The tangent vector field) $\mathbf{v}$ is said to be **parallel along** γ if $\nabla_\gamma \mathbf{v} = \mathbf{0}$ at every point of γ.

- (definition 7.4.9; p.175): The map $\Pi_\gamma^{\mathbf{pq}} : T_{\mathbf{p}}\mathcal{S} \to T_{\mathbf{q}}\mathcal{S}$ that takes $\mathbf{v}_0 \in T_{\mathbf{p}}\mathcal{S}$ to $\mathbf{v}_1 \in T_{\mathbf{q}}\mathcal{S}$ is called **parallel transport** from $\mathbf{p}$ to $\mathbf{q}$ along γ.

- (definition 8.1.1; p.179): Let $\mathcal{W}$ be the Weigarten map of an oriented surface $\mathcal{S}$ at point $\mathbf{p}$. The **Gaussian curvature** K and **mean curvature** H of $\mathcal{S}$ at $\mathbf{p}$ are defined by $K = \det(\mathcal{W})$ and $H = \frac{1}{2}\mathrm{trace}\,(\mathcal{W})$.

- (p. 187): The eigenvalues κ_1 and κ_2 of (the Weingarten map) $\mathcal{W} = \mathcal{W}_{\mathbf{p},\mathcal{S}}$ are called the **principal curvatures** of $\mathcal{S}$, and the corresponding eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are called the **principal vectors** corresponding to κ_1 and κ_2.

- (p.187): Points of a surface at which the two principal curvatures are equal are called **umbilics**.

- (definition 8.5.1; p.207): Let $\mathcal{S}$ be an orieted surface and let $\lambda \in \mathbb{R}$. The **parallel surface** $\mathcal{S}^\lambda$ of $\mathcal{S}$ is $\mathcal{S}^\lambda = \{\mathbf{p} + \lambda\,\mathbf{N}\mathbf{p} \,|\, \mathbf{p} \in \mathcal{S}\}$, where $\mathbf{N}_\mathbf{p}$ is the unit normal of $\mathcal{S}$ at the point $\mathbf{p}$.

(Geodesics)

- (definition 9.1.1; p.215): A curve γ on a surface $\mathcal{S}$ is called **geodesic** if $\ddot{\gamma}(t)$ is zero or perpendicular to the tangent plane of the surface at the point $\gamma(t)$, i.e., parallel to the unit normal, for all values of the parameter t.

- (exercise 9.1.2; p.219): A (regular) curve γ with nowhere vanishing curvature on a surface $\mathcal{S}$ is called a **pre-geodesic** on $\mathcal{S}$ if some reparametrization of γ is a geodesic on $\mathcal{S}$ (recall that a reparametrization of a geodesic is not usually a geodesic).