CARPATHIAN J. MATH. **28** (2012), No. 2, 321 - 328

Online version available at `http://carpathian.ubm.ro`
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Optimal Control on the Rotation Group $\mathsf{SO}\,(3)$

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ABSTRACT. A typical left-invariant optimal control problem on the rotation group $\mathsf{SO}\,(3)$ is investigated. The reduced Hamilton equations associated with an extremal curve are derived in a simple and elegant manner. These equations are then explicitly integrated by Jacobi elliptic functions.

1. INTRODUCTION

Invariant control systems on Lie groups provide a natural geometric setting for a variety of problems of mathematical physics, classical and quantum mechanics, elasticity, differential geometry and dynamical systems. Several (optimal control) problems related to such systems can be found, for instance, in the monographs by Jurdjevic [6], Bloch [3] or Agrachev and Sachkov [1]. In the last two decades or so, substantial work on (applied) nonlinear control has drawn attention to (left-) invariant control systems with control affine dynamics, evolving on matrix Lie groups of low dimension (see e.g. [10], [11], [13], [14] and the references therein).

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the trajectories of a given left-invariant control system, subject to appropriate boundary conditions. The application of the Maximum Principle shifts the emphasis to the language of symplectic and Poisson geometries and to the associated Hamiltonian formalism. The Maximum Principle states that the optimal solutions are projections of the extremal curves onto the base manifold. (For invariant control systems the base manifold is a Lie group G.) The extremal curves are solutions of certain Hamiltonian systems on the cotangent bundle $T^*\mathsf{G}$. The cotangent bundle $T^*\mathsf{G}$ can be realized as the direct product $\mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. As a result, each original (left-invariant) Hamiltonian induces a reduced Hamiltonian on the dual space (which comes equipped with a natural Poisson structure).

An arbitrary control affine left-invariant system on the rotation group $\mathsf{SO}\,(3)$ has the form

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{SO}\,(3),\ u \in \mathbb{R}^\ell$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{so}\,(3)$, $1 \leq \ell \leq 3$. There are four types of such systems : single-input systems with drift, two-input systems (drift-free or with drift), and fully actuated systems (i.e., drift-free three-input systems). The (non-Euclidean) elastic problem on $\mathbb{S}^2$ is associated with control systems of the first type (see [6], [5]) whereas problems related to the attitude control of a rigid body lead to optimal control problems associated with drift-free systems, underactuated or fully actuated (see [11], [16], [15]). Motion planning can be formulated as an optimal control problem associated with a control system of the third type, i.e., a two-input system with drift. In this paper we consider a typical optimal

Received: 11.12.2010; In revised form: 26.03.2012; Accepted: 23.04.2012

2010 *Mathematics Subject Classification.* 49J15, 22E60, 93B29, 33E05.

Key words and phrases. *Left-invariant control system, Pontryagin maximum principle, extremal curve, Lie-Poisson structure, elliptic function.*

control problem associated with a two-input control affine system on the rotation group $\mathsf{SO}\,(3)$. The problem is lifted, via the Pontryagin Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra $\mathfrak{so}\,(3)$. The (minus) Lie-Poisson structure on $\mathfrak{so}\,(3)^*$ can then be used to derive, in a general and elegant manner, the equations for extrema (cf. [6], [1], [8], [12], [13]). Jacobi elliptic functions are employed to derive *explicit* expressions for the extremal curves (cf. [11], [12], [13], [14]).

2. Preliminaries

2.1. **Left-invariant control systems.** Invariant control systems on Lie groups were first considered in 1972 by Brockett [4] and by Jurdjevic and Sussmann [7]. A *left-invariant control system* is a (smooth) control system evolving on some (real, finite dimensional) Lie group, whose dynamics is invariant under left translations. For the sake of convenience, we shall assume that the state space of the system is a matrix Lie group and that there are no constraints on the controls. Such a control system (evolving on G) is described as follows (cf. [6], [1])

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{2.1}$$

where the parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is a (smooth) embedding. (Here $\mathbf{1} \in \mathsf{G}$ denotes the identity matrix and $\mathfrak{g}$ denotes the Lie algebra associated with G.) An admissible control is a map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".) A *trajectory* for an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$.

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls, i.e., are of the form

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{2.2}$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Usually the elements (matrices) $B_1, \ldots, B_\ell$ are assumed to be linearly independent. Whenever $A \notin \operatorname{span}\{B_1, \ldots, B_\ell\}$, the term A is referred to as the *drift*.

2.2. **Optimal control problems.** Consider a left-invariant control system (2.1) evolving on some matrix Lie group $\mathsf{G} \leq \mathsf{GL}\,(n, \mathbb{R})$ of dimension m. In addition, it is assumed that there is a prescribed (smooth) *cost function* $L : \mathbb{R}^\ell \to \mathbb{R}$ (which is also called a Lagrangian). Let g_0 and g_1 be arbitrary but fixed points of G. We shall be interested in finding a trajectory-control pair $(g(\cdot), u(\cdot))$ which satisfies

$$g(0) = g_0, \quad g(T) = g_1 \tag{2.3}$$

and which, in addition, *minimizes* the total cost functional $\mathcal{J} = \int_0^T L(u(t))\,dt$ among all trajectories of (2.1) satisfying the same boundary conditions (2.3). The terminal time $T > 0$ can be either fixed or it can be free.

Pontryagin Maximum Principle is a necessary condition for optimality expressed most naturally in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (cf. [1], [6]). The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. The dual space $\mathfrak{g}^*$ has a natural *Poisson structure*, called the "minus Lie-Poisson structure" and given by

$$\{F, G\}_-\,(p) = -p\,([dF(p), dG(p)])$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. (Note that $dF(p)$ is a linear function on $\mathfrak{g}^*$ and hence is an element of $\mathfrak{g}$.) The Poisson manifold $(\mathfrak{g}, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. Each left-invariant Hamiltonian on the cotangent bundle $T^*\mathsf{G}$ is identified with its reduction on the dual space $\mathfrak{g}^*_-$.

To an optimal control problem (with fixed terminal time)

$$\int_0^T L(u(t))\, dt \to \min \tag{2.4}$$

subject to (2.1) and (2.3), we associate, for each real number λ and each control parameter $u \in \mathbb{R}^\ell$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\, L(u) + \xi\ (g\, \Xi(\mathbf{1}, u)) \\ &= \lambda\, L(u) + p\ (\Xi(\mathbf{1}, u))\,, \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

The Maximum Principle can be stated, in terms of the above Hamiltonians, as follows (see, e.g., [1] or [6]).

Theorem 2.1 (The Maximum Principle)**.** *Suppose the trajectory-control pair $(\bar{g}(\cdot), \bar{u}(\cdot))$ defined over the interval $[0, T]$ is a solution for the optimal control problem (2.1)-(2.3)-(2.4). Then, there exists a curve $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ with $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, T]$, and a real number $\lambda \leq 0$, such that the following conditions hold for almost every $t \in [0, T]$:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{2.5}$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{2.6}$$

$$H^\lambda_{\bar{u}(t)}\,(\xi(t)) = \max_u H^\lambda_u\,(\xi(t)) = \textit{constant}. \tag{2.7}$$

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$ defined for all $t \in [0, T]$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ defined on $[0, T]$ is said to be an *extremal pair* if $\xi(\cdot)$ is such that the conditions (2.5), (2.6) and (2.7) of the Maximum Principle hold. The projection $\xi(\cdot)$ of an extremal pair is called an *extremal*. An extremal curve is called *normal* if $\lambda = -1$ and *abnormal* if $\lambda = 0$. In this paper, we shall be concerned only with normal extremals.

If the maximum condition (2.7) eliminates the parameter u from the family of Hamiltonians (H_u), and as a result of this elimination, we obtain a smooth function H (without parameters) on $T^*\mathsf{G}$ (in fact, on $\mathfrak{g}^*_-$), then the whole (left-invariant) optimal control problem reduces to the study of trajectories of a fixed Hamiltonian vector field $\vec{H}$.

3. A left-invariant control problem on the rotation group $\mathsf{SO}(3)$

The rotation group

$$\mathsf{SO}\,(3) = \left\{a \in \mathsf{GL}\,(3, \mathbb{R})\ :\ a^\top a = \mathbf{1},\ \det a = 1\right\}$$

is a three-dimensional compact and connected matrix Lie group. The associated Lie algebra is given by

$$\mathfrak{so}\,(3) = \left\{ \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\ :\ a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{so}\,(3)$. (The bracket operation is given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$ and $[E_1, E_2] = E_3$.) We shall identify $\mathfrak{so}\,(3)$ with (the cross-product Lie algebra) $\mathbb{R}^3_\wedge$. Consider the following optimal control problem

$$\dot{g} = g\left(E_3 + u_1 E_1 + u_2 E_2\right), \quad g \in \mathsf{SO}\,(3),\ u = (u_1, u_2) \in \mathbb{R}^2 \tag{3.8}$$

$$g(0) = g_0, \quad g(T) = g_1 \quad (g_0, g_1 \in \mathsf{SO}\,(3)) \tag{3.9}$$

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t)\right)\, dt \to \min. \tag{3.10}$$

This problem appears in optimal path planning; it is also related to a variation of the classical elastic problem (cf. [5], [6],[1]).

We will identify $\mathfrak{so}\,(3)^*$ with $\mathfrak{so}\,(3)$ via the pairing

$$\left\langle \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Then each extremal curve $p(\cdot)$ is identified with a curve $P(\cdot)$ in $\mathfrak{so}\,(3)$ via the formula $\langle P(t), A\rangle = p(t)(A)$ for all $A \in \mathfrak{so}\,(3)$. Thus

$$P(t) = \begin{bmatrix} 0 & -P_3(t) & P_2(t) \\ P_3(t) & 0 & -P_1(t) \\ -P_2(t) & P_1(t) & 0 \end{bmatrix} \tag{3.11}$$

where $P_i(t) = \langle P(t), E_i\rangle = p(t)(E_i), \quad i = 1, 2, 3.$

Now consider a Hamiltonian H on the (minus) Lie-Poisson structure for $\mathfrak{so}\,(3)^*$. The equations of motion take the form

$$\dot{p}_i = -p\left([E_i, dH(p)]\right), \quad i = 1, 2, 3$$

or, explicitly,

$$\dot{p}_1 = \frac{\partial H}{\partial p_3}\, p_2 - \frac{\partial H}{\partial p_2}\, p_3 \tag{3.12}$$

$$\dot{p}_2 = \frac{\partial H}{\partial p_1}\, p_3 - \frac{\partial H}{\partial p_3}\, p_1 \tag{3.13}$$

$$\dot{p}_3 = \frac{\partial H}{\partial p_2}\, p_1 - \frac{\partial H}{\partial p_1}\, p_2. \tag{3.14}$$

We note that $C : \mathfrak{so}\,(3)^* \to \mathbb{R}$, $C(p) = p_1^2 + p_2^2 + p_3^2$ is a Casimir function. The following result is not hard to prove (cf. [8]).

Proposition 3.1. *For the left-invariant optimal control problem (3.8)-(3.9)-(3.10), the extremal control* $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ *is given by* $\bar{u}_1 = \frac{1}{c_1}p_1$ *and* $\bar{u}_2 = \frac{1}{c_2}p_2$*, where*

$$H(p) = \frac{1}{2}\left(\frac{1}{c_1}p_1^2 + \frac{1}{c_2}p_2^2\right) + p_3$$

and

$$\dot{p}_1 = -\frac{1}{c_2} p_2 p_3 + p_2 \tag{3.15}$$

$$\dot{p}_2 = \frac{1}{c_1} p_1 p_3 - p_1 \tag{3.16}$$

$$\dot{p}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1}\right) p_1 p_2. \tag{3.17}$$

Remark 3.1. The extremal trajectories (i.e., the solution curves of the reduced Hamilton equations) are intersections of quadric surfaces $\frac{1}{c_1} p_1^2 + \frac{1}{c_2} p_2^2 + 2p_3 = \text{const}$ and spheres $p_1^2 + p_2^2 + p_3^2 = \text{const}$.

4. Integration by Jacobi elliptic functions

Given a number $k \in [0,1]$, the function $F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$ is called an (incomplete) *elliptic integral* of the first kind. The parameter k is known as the modulus. The inverse function $\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is called the amplitude, from which the basic *Jacobi elliptic functions* are derived :

$$\begin{aligned}
\mathrm{sn}(x,k) &= \sin \mathrm{am}(x,k) \quad \text{(sine amplitude)} \\
\mathrm{cn}(x,k) &= \cos \mathrm{am}(x,k) \quad \text{(cosine amplitude)} \\
\mathrm{dn}(x,k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x,k)} \quad \text{(delta amplitude)}.
\end{aligned}$$

(For the degenerate cases $k = 0$ and $k = 1$, we recover the circular functions and the hyperbolic functions, respectively.) Nine other elliptic functions are defined by taking reciprocals and quotients; in particular, we get

$$\mathrm{dc}(\cdot, k) = \frac{\mathrm{dn}(\cdot,k)}{\mathrm{cn}(\cdot,k)}, \ \mathrm{ns}(\cdot,k) = \frac{1}{\mathrm{sn}(\cdot,k)}, \ \mathrm{nc}(\cdot,k) = \frac{1}{\mathrm{cn}(\cdot,k)}, \ \mathrm{ds}(\cdot,k) = \frac{\mathrm{dn}(\cdot,k)}{\mathrm{sn}(\cdot,k)}.$$

Simple elliptic integrals can be expressed in terms of the appropriate inverse (elliptic) functions. The following formulas hold true for $b < a \le x$ (see e.g. [2]):

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a} \mathrm{dc}^{-1}\left(\frac{x}{a}, \frac{b}{a}\right) \tag{4.18}$$

$$\int_x^{\infty} \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a} \mathrm{ns}^{-1}\left(\frac{x}{a}, \frac{b}{a}\right) \tag{4.19}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 + b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \mathrm{nc}^{-1}\left(\frac{x}{a}, \frac{b}{\sqrt{a^2 + b^2}}\right) \tag{4.20}$$

$$\int_x^{\infty} \frac{dt}{\sqrt{(t^2 - a^2)(t^2 + b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \mathrm{ds}^{-1}\left(\frac{x}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right). \tag{4.21}$$

When the coefficients c_1 and c_2 are equal, then a straightforward computation gives explicit formulas in terms of circular functions (cf. [14]).

Proposition 4.2. *When* $c = c_1 = c_2$, *the reduced Hamilton equations* (3.15)*-*(3.16)*-*(3.17) *have the solutions*

$$\begin{aligned} p_1(t) &= \sqrt{k_1}\cos\left(\left(1-\frac{k_2}{c}\right)t+k_3\right) \\ p_2(t) &= \sqrt{k_1}\sin\left(\left(1-\frac{k_2}{c}\right)t+k_3\right) \\ p_3(t) &= k_2, \end{aligned}$$

where $k_1 = p_1^2(0)+p_2^2(0)$, $k_2 = p_3(0)$ *and* $k_3 = -\tan^{-1}\dfrac{p_2(0)}{p_1(0)}\cdot$

In the generic case, when the coefficients c_1 and c_2 are distinct, Jacobi elliptic functions are employed. Recall that H and C denote the reduced Hamiltonian and the Casimir function, respectively.

Proposition 4.3. *Let* $p(\cdot)$ *be an integral curve of* $\vec{H}$ *such that* $H(p(0))=h_0$, $C(p(0)) = c_0 > 0$ *and* $h_0^2 > c_0$. *Assume that* $c_2 < c_1$. *Then*

$$\begin{aligned} p_1(t) &= \pm\sqrt{\frac{c_1}{c_2-c_1}\left(2c_2h_0-c_0-2c_2p_3(t)+p_3^2(t)\right)} \\ p_2(t) &= \pm\sqrt{\frac{c_2}{c_1-c_2}\left(2c_1h_0-c_0-2c_1p_3(t)+p_3^2(t)\right)} \\ p_3(t) &= \frac{\alpha-\beta\,\phi(t)}{1-\phi(t)}\cdot \end{aligned}$$

(i) If $h_0-c_1<-\sqrt{h_0^2-c_0}$, $\sqrt{h_0^2-c_0}<h_0-c_2$ *and* $\dfrac{h_0-c_1+\sqrt{h_0^2-c_0}}{h_0-c_1-\sqrt{h_0^2-c_0}}<\dfrac{h_0-c_2+\sqrt{h_0^2-c_0}}{h_0-c_2-\sqrt{h_0^2-c_0}}$, *then* $\phi(t)$ *takes one of the following forms*

$$a\cdot\mathrm{ns}\left(\Omega t,\frac{b}{a}\right)\quad \textit{or}\quad a\cdot\mathrm{dc}\left(\Omega t,\frac{b}{a}\right);$$

here,

$$a^2 = \frac{h_0-c_2+\sqrt{h_0^2-c_0}}{h_0-c_2-\sqrt{h_0^2-c_0}}\quad \textit{and}\quad b^2 = \frac{h_0-c_1+\sqrt{h_0^2-c_0}}{h_0-c_1-\sqrt{h_0^2-c_0}}\cdot$$

(ii) If $-\sqrt{h_0^2-c_0} < h_0-c_1 < \sqrt{h_0^2-c_0}$ *and* $\sqrt{h_0^2-c_0} < h_0-c_2$, *then* $\phi(t)$ *takes one of the following forms*

$$a\cdot\mathrm{nc}\left(\Omega\sqrt{a^2+b^2}\,t,\frac{b}{\sqrt{a^2+b^2}}\right)\quad \textit{or}\quad \sqrt{a^2+b^2}\cdot\mathrm{ds}\left(\Omega\sqrt{a^2+b^2}\,t,\frac{b}{\sqrt{a^2+b^2}}\right);$$

here,

$a^2 = \dfrac{h_0-c_2+\sqrt{h_0^2-c_0}}{h_0-c_2-\sqrt{h_0^2-c_0}}$ *and* $b^2 = \dfrac{h_0-c_1+\sqrt{h_0^2-c_0}}{h_0-c_1-\sqrt{h_0^2-c_0}}\cdot$ *(Throughout,* $\alpha = h_0+\sqrt{h_0^2-c_0}$, $\beta=h_0-\sqrt{h_0^2-c_0}$, $M=-\dfrac{(h_0-c_2-\sqrt{h_0^2-c_0})(h_0-c_1-\sqrt{h_0^2-c_0})}{4(h_0^2-c_0)}$ *and* $\Omega = \dfrac{(\alpha-\beta)M}{\sqrt{c_1c_2}}\cdot$*)*
Similar formulas hold for the case $c_1 < c_2$.

Proof. Assume that $c_2 < c_1$. We get

$$p_1^2 = \frac{c_1}{c_2 - c_1}\left(2c_2h_0 - c_0 - 2c_2p_3 + p_3^2\right)$$
$$p_2^2 = \frac{c_2}{c_1 - c_2}\left(2c_1h_0 - c_0 - 2c_1p_3 + p_3^2\right)$$

and thus

$$\dot{p}_3^2 = \frac{1}{c_1c_2}\left(c_0 - 2c_2h_0 + 2c_2p_3 - p_3^2\right)\left(2c_1h_0 - c_0 - 2c_1p_3 + p_3^2\right). \tag{4.22}$$

The right-hand side of this equation can be written as

$$\frac{1}{c_1c_2}\left(\mu_1(p_3-\alpha)^2 + \nu_1(p_3-\beta)^2\right)\left(\mu_2(p_3-\alpha)^2 + \nu_2(p_3-\beta)^2\right)$$

where

$$\mu_1 = \frac{h_0 - c_2 - \sqrt{h_0^2 - c_0}}{2\sqrt{h_0^2 - c_0}} \qquad \mu_2 = \frac{c_1 - h_0 + \sqrt{h_0^2 - c_0}}{2\sqrt{h_0^2 - c_0}}$$
$$\nu_1 = \frac{c_2 - h_0 - \sqrt{h_0^2 - c_0}}{2\sqrt{h_0^2 - c_0}} \qquad \nu_2 = \frac{h_0 - c_1 + \sqrt{h_0^2 - c_0}}{2\sqrt{h_0^2 - c_0}}$$
$$\alpha = h_0 + \sqrt{h_0^2 - c_0} \qquad \beta = h_0 - \sqrt{h_0^2 - c_0}.$$

Denote $\sqrt{\mu_1\mu_2}$ by M. Now, straightforward algebraic manipulation as well as simple integration and appropriate change of variables yield explicit expressions for the solutions of the differential equation (4.22). We get

$$p_3(t) = \frac{\alpha - \beta a\cdot \mathrm{dc}\left(\frac{(\alpha-\beta)M}{\sqrt{c_1c_2}}at,\ \frac{b}{a}\right)}{1 - a\cdot \mathrm{dc}\left(\frac{(\alpha-\beta)M}{\sqrt{c_1c_2}}at,\ \frac{b}{a}\right)}$$

(corresponding to the integral (4.18)) or

$$p_3(t) = \frac{\alpha - \beta a\cdot \mathrm{ns}\left(\frac{(\alpha-\beta)M}{\sqrt{c_1c_2}}at,\ \frac{b}{a}\right)}{1 - a\cdot \mathrm{ns}\left(\frac{(\alpha-\beta)M}{\sqrt{c_1c_2}}at,\ \frac{b}{a}\right)}$$

(corresponding to the integral (4.19)), where

$$a^2 = \frac{h_0 - c_2 + \sqrt{h_0^2 - c_0}}{h_0 - c_2 - \sqrt{h_0^2 - c_0}} \quad \text{and} \quad b^2 = \frac{h_0 - c_1 + \sqrt{h_0^2 - c_0}}{h_0 - c_1 - \sqrt{h_0^2 - c_0}}.$$

Conditions $a^2 > 0$, $b^2 > 0$ and $b < a$ translate into $h_0 - c_1 < -\sqrt{h_0^2 - c_0}$, $\sqrt{h_0^2 - c_0} < h_0 - c_2$ and

$$\frac{h_0 - c_1 + \sqrt{h_0^2 - c_0}}{h_0 - c_1 - \sqrt{h_0^2 - c_0}} < \frac{h_0 - c_2 + \sqrt{h_0^2 - c_0}}{h_0 - c_2 - \sqrt{h_0^2 - c_0}}.$$

Alternatively, we get

$$p_3(t) = \frac{\alpha - \beta a\cdot \mathrm{nc}\left(\frac{(\alpha-\beta)M\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}t,\ \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - a\cdot \mathrm{nc}\left(\frac{(\alpha-\beta)M\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}t,\ \frac{b}{\sqrt{a^2+b^2}}\right)}$$

(corresponding to the integral (4.20)) or

$$p_3(t) = \frac{\alpha - \beta\sqrt{a^2+b^2}\cdot \mathrm{ds}\left(\frac{(\alpha-\beta)M\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}t,\ \frac{b}{\sqrt{a^2+b^2}}\right)}{1-\sqrt{a^2+b^2}\cdot \mathrm{ds}\left(\frac{(\alpha-\beta)M\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}t,\ \frac{b}{\sqrt{a^2+b^2}}\right)}$$

(corresponding to the integral (4.21)), where a^2 and b^2 have the same expression as above. The constraints now translate into $-\sqrt{h_0^2-c_0} < h_0 - c_1 < \sqrt{h_0^2-c_0}$ and $\sqrt{h_0^2-c_0} < h_0 - c_2$. In the same fashion, similar formulas can be derived for the case when $c_1 < c_2$. □

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