MATRIX LIE GROUPS

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TALK OUTLINE

1. What is a matrix Lie group ?

2. Matrices revisited.

3. Examples of matrix Lie groups.

4. Matrix Lie algebras.

5. A glimpse at elementary Lie theory.

6. Life beyond elementary Lie theory.

1. What is a matrix Lie group ?

- **Matrix Lie groups** are groups of invertible matrices that have desirable geometric features. So matrix Lie groups are simultaneously *algebraic* and *geometric* objects.

- Matrix Lie groups naturally arise in
  - *geometry* (classical, algebraic, differential)
  - *complex analyis*
  - *differential equations*
  - *Fourier analysis*
  - *algebra* (group theory, ring theory)
  - *number theory*
  - *combinatorics*.

- Matrix Lie groups are encountered in many applications in
  - *physics* (geometric mechanics, quantum control)
  - *engineering* (motion control, robotics)
  - *computational chemistry* (molecular motion)
  - *computer science* (computer animation, computer vision, quantum computation).

- "It turns out that **matrix** [Lie] **groups pop up in virtually any investigation of objects with symmetries**, such as molecules in chemistry, particles in physics, and projective spaces in geometry". (K. Tapp, 2005)

- EXAMPLE 1 : The *Euclidean group* E (2).

  $$\mathsf{E}\,(2) = \left\{ F : \mathbb{R}^2 \to \mathbb{R}^2 \,|\, F \text{ is an isometry} \right\}.$$

  The vector space $\mathbb{R}^2$ is equipped with the *standard Euclidean structure* (the "dot product")

  $$x \bullet y = x_1 y_1 + x_2 y_2 \quad (x, y \in \mathbb{R}^2),$$

  hence with the *Euclidean distance*

  $$d_E(x, y) = \sqrt{(y - x) \bullet (y - x)} \quad (x, y \in \mathbb{R}^2).$$

Any isometry F of (the Euclidean plane) $\mathbb{R}^2$ has the form

$$F(x) = Ax + c,$$

where $A \in \mathbb{R}^{2\times 2}$ such that $A^\top A = I_2$, and $c \in \mathbb{R}^2 = \mathbb{R}^{2\times 1}$.

It turns out that the Euclidean group $\mathsf{E}(2)$ is (isomorphic to) the group of invertible matrices

$$\left\{ \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \mid A \in \mathbb{R}^{2\times 2},\, A^\top A = I_2,\, c \in \mathbb{R}^2 \right\}.$$

The underlying set of this group of matrices consists of two components, each of which is (diffeomorphic to) the *smooth submanifold*

$$\mathbb{R}^2 \times \mathbb{S}^1 \subset \mathbb{R}^4.$$

- EXAMPLE 2 : The *rotation group* SO (2).

  A non-identity isometry of (the Euclidean plane) $\mathbb{R}^2$ is *exactly* one of the following:

  – rotation

  – translation

  – reflection

  – glide reflection.

  The **rotation group** SO (2) consists of all isometries of $\mathbb{R}^2$ which

  – fix the origin (i.e., are linear transformations)

  – preserve the orientation (of $\mathbb{R}^2$).

It turns out that the rotation group $\mathsf{SO}(2)$ is (isomorphic to) a group of invertible matrices

$$\begin{aligned} \mathsf{SO}(2) &= \left\{ A \in \mathbb{R}^{2\times 2} \mid A^\top A = I_2 \right\} \\ &= \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\}. \end{aligned}$$

The underlying set of this group of matrices is (diffeomorphic to) the *smooth submanifold*

$$\mathbb{S}^1 = \{ z \in \mathbb{C} \mid \ |z| = 1 \} \subset \mathbb{C} = \mathbb{R}^2.$$

- EXAMPLE 3 : The 3-sphere $\mathbb{S}^3$.

  $\mathbb{S}^3 = \left\{ (x_0, x_1, x_2, x_3) \,|\, x_0^2 + x_1^2 + x_3^2 + x_4^2 = 1 \right\} \subset \mathbb{R}^4$

  is a *smooth submanifold*.

  It is also a *group*. Indeed, the vector space $\mathbb{R} \times \mathbb{R}^3$, equipped with the product

  $$\mathbf{p} \cdot \mathbf{q} = (p_0 q_0 - p \bullet q, p_0 q + q_0 p + p \times q) ,$$

  where $\mathbf{p} = (p_0, p)$, $\mathbf{q} = (q_0, q) \in \mathbb{R} \times \mathbb{R}^3$, is a (real) *division algebra* (or skew field), denoted by $\mathbb{H}$; its elements are called **quaternions** (or Hamilton numbers).

  $\mathbb{S}^3$ is exactly the *group of unit quaternions* :

  $$\mathbb{S}^3 = \{ \mathbf{x} = (x_0, x) \,| \; |\mathbf{x}| = 1 \} \subset \mathbb{R} \times \mathbb{R}^3.$$

  (The *modulus* of the quaternion $\mathbf{x} = (x_0, x) \in \mathbb{H}$ is $|\mathbf{x}| = \sqrt{x_0^2 + \|x\|^2}$.)

It turns out that the 3-sphere $\mathbb{S}^3$, as a group, is isomorphic to a group of invertible matrices

$$\mathbb{S}^3 = \left\{ \begin{bmatrix} x_0 + ix_1 & -(x_2 + ix_3) \\ x_2 - ix_3 & x_0 - ix_1 \end{bmatrix} \mid x_0^2 + \|x\|^2 = 1 \right\}$$

$$= \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

This group is the **special unitary group** SU(2); it can be viewed as representing all *unitary transformations* of the *standard Hermitian space* $\mathbb{C}^2$, of determinant equal to 1.

(The *standard Hermitian product* on $\mathbb{C}^2$ is defined by

$$z \odot w = \bar{z}_1 w_1 + \bar{z}_2 w_2,$$

where $z = (z_1, z_2)$, $w = (w_1, w_2) \in \mathbb{C}^2$.)

- DEFINITION : A **matrix Lie group** is any *subgroup* G of the general linear group

$$\mathsf{GL}(n, \Bbbk), \quad \Bbbk \in \{\mathbb{R}, \mathbb{C}\}$$

(for some positive integer n) which is also a *smooth submanifold* of the matrix space $\Bbbk^{n \times n}$.

- REMARK 1 : As their name suggests, matrix Lie groups are (abstract) *Lie groups*. Though not all Lie groups are (isomorphic to) matrix Lie groups, most of the interesting examples are.

- REMARK 2 : A matrix Lie group G is a *closed subset* of $\mathsf{GL}(n, \Bbbk) \subset \Bbbk^{n \times n}$. Quite remarkably - and this is an important result in the theory of Lie groups - it turns out that **any closed subgroup of** $\mathsf{GL}(n, \Bbbk)$ **is a matrix Lie group**.

- A subset S of the Euclidean space $\mathbb{R}^m$ is called a **smooth submanifold**, of dimension ℓ, provided that one (and hence all) of the following equivalent conditions are satisfied :

  (a) For every $x \in S$, there exist a nbd U of x and a smooth *diffeomorphism* $\phi : U \to \tilde{U} \subseteq \mathbb{R}^m$ such that

  $$\phi(S \cap U) = \tilde{U} \cap \mathbb{R}^\ell.$$

  (b) For every $x \in S$, there exist a nbd U of x and a smooth *submersion* $F : U \to \mathbb{R}^{m-\ell}$ such that

  $$S \cap U = F^{-1}(0).$$

(c) For every $x \in S$, there exist a nbd U of $x = (x_1, \ldots, x_m)$, a nbd U' of $x' = (x_1, \ldots, x_\ell)$ and smooth *functions* $h_i : U' \to \mathbb{R}$, $i = 1, \ldots, m - \ell$ such that, possibly after a permutation of coordinates,

$$S \cap U = \text{graph}\,(H),$$

where $H = (h_1, \ldots, h_{m-\ell}) : U' \to \mathbb{R}^{m-\ell}$.

(d) For every $x \in S$, there exist a nbd U of x, a nbd V of $0 \in \mathbb{R}^\ell$ and a smooth *embedding* $\Phi : V \to \mathbb{R}^m$ such that

$$S \cap U = \text{im}\,(\Phi).$$

- In (b) we think of a smooth submanifold as the *zero-set* of a smooth submersion, in (c) as a *graph* of a smooth map, and in (d) as a *parametrized set*. All these are local descriptions.

- A (real) **Lie group** is a *smooth manifold* which is also a *group* so that the group operations are smooth.

- In most of the literature, Lie groups are defined to be *real analytic*. In fact, no generality is lost by this more restrictive definition. *Smooth Lie groups always support an analytic group structure*, and something even stronger is true.

  HILBERT's 5th PROBLEM was to show that *if* G *is only assumed to be a topological manifold with continuous group operations, then it is, in fact, a real analytic Lie group*. (This was finally proved by the combined work of A. GLEASON, D. MONTGOMERY, L. ZIPPIN and H. YAMABE in 1952.)

- A **Lie subgroup** of a Lie group G is a Lie group H that is an *abstract subgroup* and an *immersed submanifold* of G.

  This means that the inclusion map $\iota : H \to G$ is a one-to-one *smooth immersion*; when ι is a smooth embedding, then the set H is *closed* in G.

- FACT : Any closed abstract subgroup H of a Lie group G has a unique smooth structure which makes it into a Lie subgroup of G; in particular, H has the induced topology.

  Matrix Lie groups are closed Lie subgroups of general linear groups. They are also known in literature as *closed linear (Lie) groups*.

2. **Matrices revisited.**

- **Matrices** (and groups of matrices) have been introduced by
  - Arthur CAYLEY (1821-1895)
  - William Rowan HAMILTON (1805-1865)
  - James Joseph SYLVESTER (1814-1897)

  in the 1850s.

- About twenty years later, Sophus LIE (1842-1899) began his research on *"continuous groups of transformations"*, which gave rise to the modern theory of the so-called **Lie groups**.

- In 1872, Felix KLEIN (1849-1925) delivered his inaugural address in Erlangen in which he gave a very general view on what geometry should be regarded as; this lecture soon became known as **"The Erlanger Programm"**. KLEIN saw *geometry* as

  the study of invariants under a group of transformations.

  LIE's and KLEIN's research was to a certain extent inspired by their deep interest in the *theory of groups* and in various aspects of the notion of *symmetry*.

- The ground field is $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$. Consider the **matrix space** $\mathbb{k}^{n \times n}$ of all $n \times n$ matrices over $\mathbb{k}$.

  The set $\mathbb{k}^{n \times n}$ has a certain amount of structure :

  - $\mathbb{k}^{n \times n}$ is a *real vector space*.
  - $\mathbb{k}^{n \times n}$ is a *normed algebra*.
  - $\mathbb{k}^{n \times n}$ is a *Lie algebra*.

(a) $\mathbb{k}^{n\times n}$ is a *real vector space*.

Firstly, the correspondence

$$A = \left[a_{ij}\right] \mapsto \mathbf{a} = (a_{11}, \cdots, a_{n1}, a_{12}, \cdots, a_{nn})$$

defines an isomorphism between (vector spaces over $\mathbb{k}$) $\mathbb{k}^{n\times n}$ and $\mathbb{k}^{n^2}$.

Secondly, since any vector space over $\mathbb{C}$ can be viewed as a vector space over $\mathbb{R}$ (of twice the dimension), we have that $\mathbb{C}^{n^2}$, as a real vector space, is isomorphic to $\mathbb{R}^{2n^2}$:

$$\mathbb{C}^{n^2} = \mathbb{R}^{n^2} \oplus i\,\mathbb{R}^{n^2} = \mathbb{R}^{2n^2}.$$

So we have

$$\mathbb{k}^{n\times n} = \mathbb{k}^{n^2} = \begin{cases} \mathbb{R}^{n^2} & \text{if } \mathbb{k} = \mathbb{R} \\ \mathbb{R}^{2n^2} & \text{if } \mathbb{k} = \mathbb{C}. \end{cases}$$

The matrix space $\mathbb{k}^{n\times n}$ inherits a *natural (metric) topology*, induced by the Euclidean distance on $\mathbb{k}^{n^2}$.

For a matrix $A = \left[a_{ij}\right] \in \mathbb{k}^{n\times n}$, the (induced) norm

$$\begin{aligned} \|A\|_F &= \|\mathbf{a}\|_2 = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} \\ &= \sqrt{\operatorname{tr}(A^*A)} \end{aligned}$$

is called the *Frobenius norm*.

(b) $\mathbb{k}^{n\times n}$ is a *normed algebra*.

To any matrix $A \in \mathbb{k}^{n\times n}$ there corresponds a *linear endomorphism* $x \mapsto Ax$ of (the vector space) $\mathbb{k}^n$; conversely, any such endomorphism is defined by an $n \times n$ matrix over $\mathbb{k}$.

The correspondence

$$A = \left[a_{ij}\right] \in \mathbb{k}^{n\times n} \mapsto \left(x \mapsto Ax = (\sum_{j=1}^{n} a_{ij}x_j)\right)$$

is an isomorphism between (associative algebras over $\mathbb{k}$) $\mathbb{k}^{n\times n}$ and $\mathsf{End}\,(\mathbb{k}^n)$.

(We identify $n \times n$ matrices with linear endomorphisms of $\mathbb{k}^n$ without change of notation.)

The associative algebra $\mathsf{End}\,(\mathbb{k}^n)$ has a natural metric topology, generated by the so-called *operator norm* :

$$\begin{aligned} \|A\| &= \max_{\|x\|_2 = 1} \|Ax\|_2 \\ &= \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}, \end{aligned}$$

where $\|\cdot\|_2$ is the 2-norm (or the Euclidean norm) on $\mathbb{k}^n$.

The operator norm $\|\cdot\|$ and the Frobenius norm $\|\cdot\|_F$ are both *sub-multiplicative* norms (on $\mathbb{k}^{n\times n}$), but they are not equal (for instance, $\|I_n\| = 1$ whereas $\|I_n\|_F = \sqrt{n}$).

Being defined on the same finite-dimensional vector space, these (matrix) norms are *equivalent* (i.e., they generate the same metric topology).

(c) $\mathbb{k}^{n\times n}$ is a *Lie algebra*.

The *matrix commutator*

$$[A,B] = AB - BA \qquad (A,B \in \mathbb{k}^{n\times n})$$

defines on the matrix space $\mathbb{k}^{n\times n}$ a Lie algebra structure.

When viewed as a Lie algebra, $\mathbb{k}^{n\times n}$ is usually denoted by $\mathfrak{gl}\,(n,\mathbb{k})$.

Any element (matrix) $A \in \mathbb{k}^{n\times n}$ defines two different *vector fields* on $\mathbb{k}^{n\times n} = \mathbb{k}^{n^2}$:

$$X \mapsto AX \quad \text{and} \quad X \mapsto XA.$$

- DEFINITION : A **Lie algebra** over $\mathbb{k}$ is a vector space $\mathfrak{A}$ over $\mathbb{k}$ equipped with a bilinear multiplication

$$[\cdot,\cdot] : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A},$$

called the *Lie bracket*, which satisfy the following conditions :

(a) skew-symmetry :

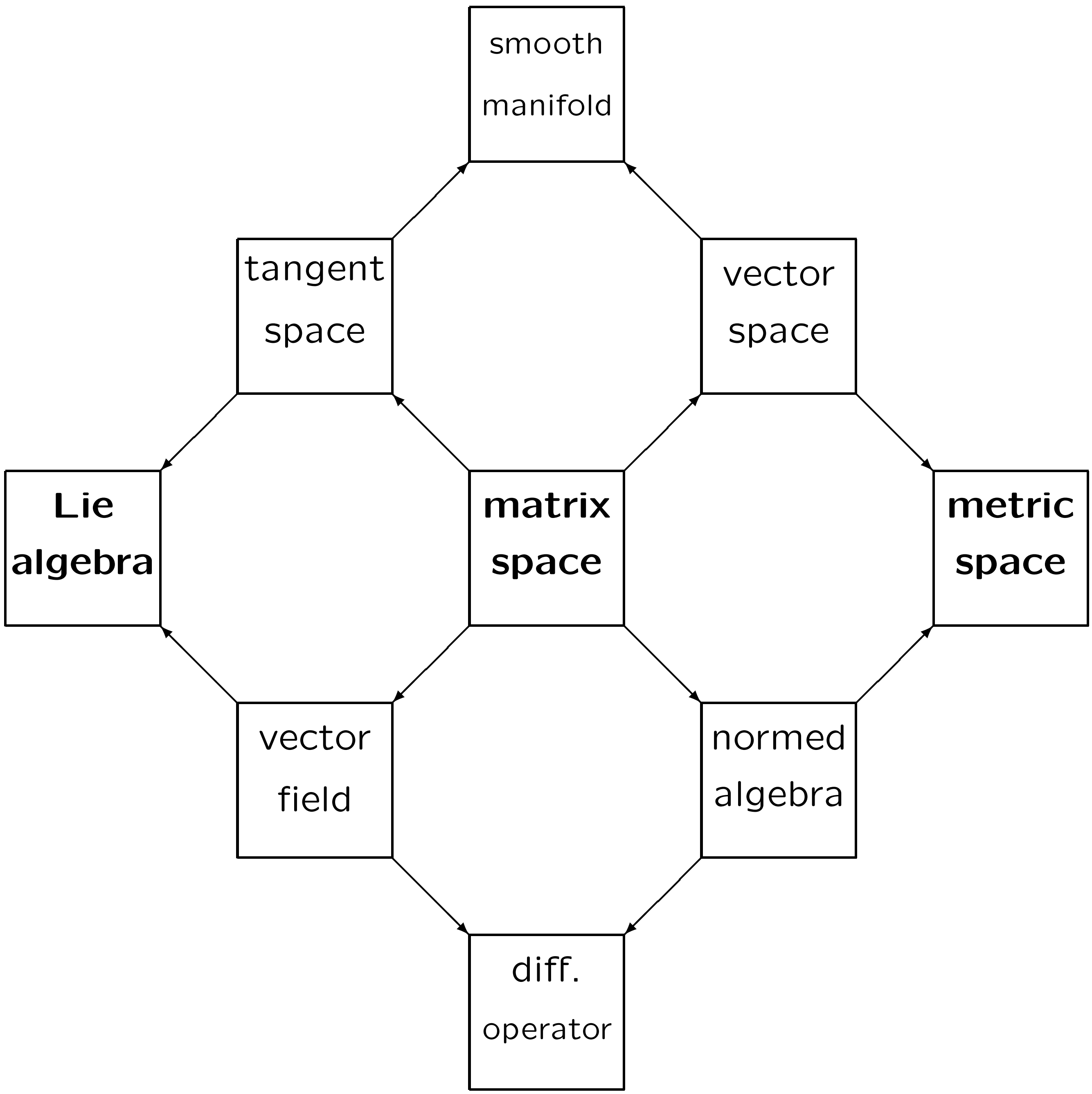
$$[x,y] = -[y,x] \qquad (x,y \in \mathfrak{A}).$$

(b) the JACOBI identity :

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad (x,y,z \in \mathfrak{A}).$$

A Lie algebra is an algebraic structure whose main use is in studying geometric objects such as *Lie groups* and *homogeneous spaces*. The term "Lie algebra" was introduced by Hermann WEYL (1885-1955) in the 1930s.

The matrix space $\mathbb{k}^{n\times n}$



- The map

$$\rho_n : \mathbb{C}^{n\times n} \to \mathbb{R}^{2n\times 2n}, \quad Z = X + iY \mapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$$

has the following properties :

(a) For all $A \in \mathbb{C}^{n\times n}$, the diagram

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{R}^{2n} \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \rho_n(A)} \\ \mathbb{C}^n & \longrightarrow & \mathbb{R}^{2n} \end{array}$$

commutes.

(b) For all $\lambda \in \mathbb{R}$ and $A, B \in \mathbb{C}^{n\times n}$,

- $\rho_n(\lambda A) = \lambda \rho_n(A)$.
- $\rho_n(A + B) = \rho_n(A) + \rho_n(B)$.
- $\rho_n(AB) = \rho_n(A)\rho_n(B)$.

ρ_n is an injective homomorphism of algebras (over $\mathbb{R}$).

We identify $\mathbb{C}^{n\times n}$ with $\rho_n(\mathbb{C}^{n\times n}) \subset \mathbb{R}^{2n\times 2n}$.

- Consider the **general linear group**

$$\mathsf{GL}\,(n,\mathbb{k}) = \left\{ g \in \mathbb{k}^{n\times n} \mid \det g \neq 0 \right\}.$$

$\mathsf{GL}\,(n,\mathbb{k})$ is an *open subset* of the matrix space $\mathbb{k}^{n\times n}$ and so, in particular, is a *smooth manifold*. (In fact, $\mathsf{GL}\,(n,\mathbb{k})$ is a *Lie group*.)

We identify the complex general linear group $\mathsf{GL}\,(n,\mathbb{C})$ with the subgroup

$$\begin{aligned} \rho_n(\mathsf{GL}\,(n,\mathbb{C})) &= \left\{ \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \mid \det(X+iY) \neq 0 \right\} \\ &= \{ g \in \mathsf{GL}\,(2n,\mathbb{R}) \mid g\,\mathbb{J} = \mathbb{J}\,g \} \end{aligned}$$

of the general linear group $\mathsf{GL}\,(2n,\mathbb{R})$.

(Here $\mathbb{J} = J_{n,n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.)

- For any $A \in \mathbb{k}^{n \times n}$, the *matrix exponential*

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

is defined; it turns out that e^A is nonsingular.

The **exponential map**

$$\exp : \mathbb{k}^{n \times n} \to \mathsf{GL}\,(n, \mathbb{k}), \quad A \mapsto e^A$$

has the following properties :

(a) exp is a smooth map (in fact, it is locally a diffeomorphism at 0).

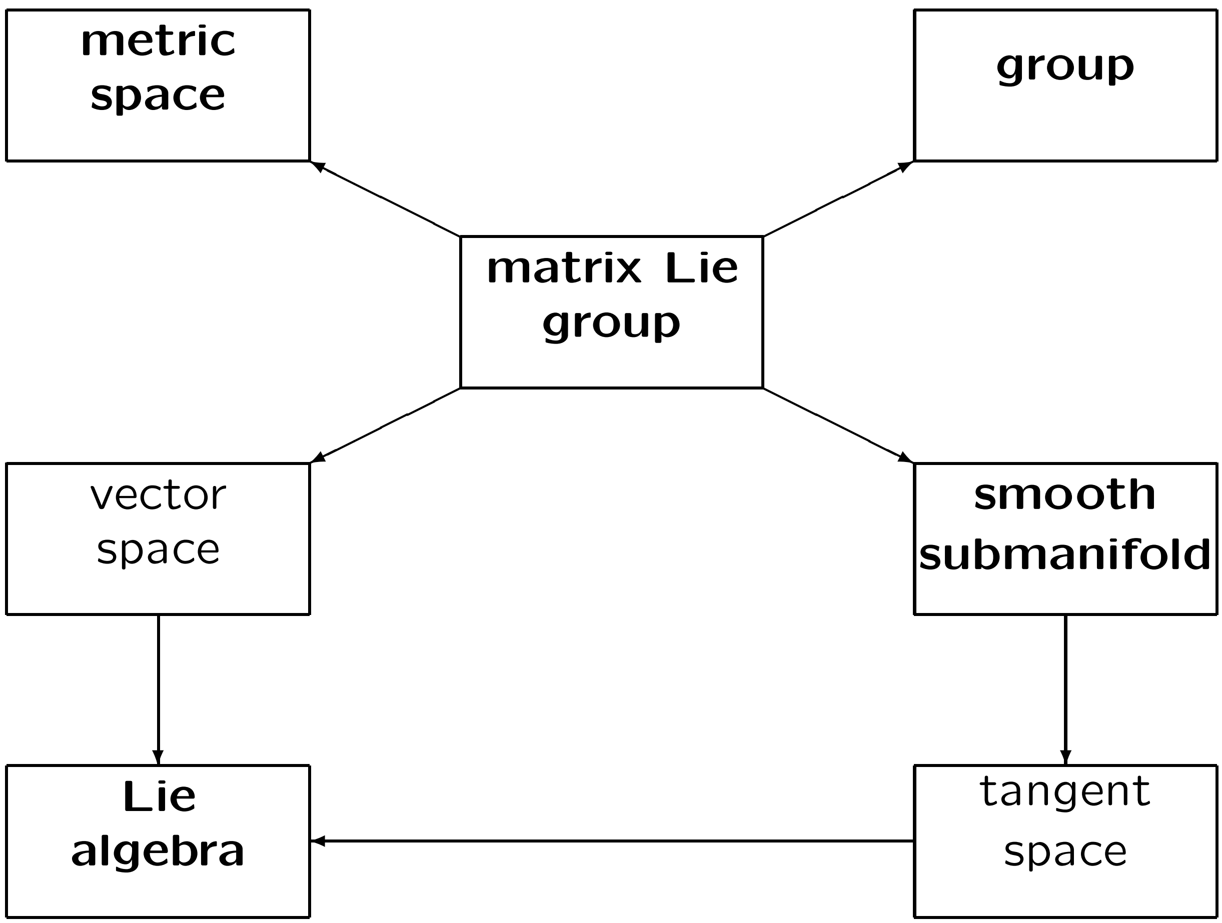
(b) $e^X e^Y = e^{X+Y}$ if X and Y commute.

(c) $t \mapsto e^{tX}$ is a smooth curve in $\mathsf{GL}\,(n, \mathbb{k})$ that is $1 = I_n$ at $t = 0$.

(d) $\frac{d}{dt}\left(e^{tX}\right) = Xe^{tX}$.

(e) $\det e^X = e^{\mathrm{tr}\,(X)}$.

The matrix Lie group $\mathsf{G} \leq \mathsf{GL}(n, \mathbb{k})$



3. Examples of matrix Lie groups.

(Again, the ground field is $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$.)

- The *additive* group $\Bbbk$:

$$\Bbbk \cong \left\{ \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \mid \alpha \in \Bbbk \right\} \leq \mathsf{GL}\,(2, \Bbbk).$$

- The *multiplicative* group $\Bbbk^\times$:

$$\Bbbk^\times = \mathsf{GL}\,(1, \Bbbk).$$

- The *circle* (or 1-dimensional torus) $\mathbb{S}^1$:

$$\begin{aligned} \mathbb{S}^1 &= \{ z \in \mathbb{C} \mid \ |z| = 1 \} \\ &= \left\{ \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \mid \alpha^2 + \beta^2 = 1 \right\} \leq \mathsf{GL}\,(2, \mathbb{R}). \end{aligned}$$

(In fact, $\mathbb{S}^1 \cong \mathsf{SO}\,(2)$.)

FACT : The *direct product* of two matrix Lie groups is a matrix Lie group.

- The *vector Lie group* $\mathbb{k}^n$ (the direct product of n copies of the additive group $\mathbb{k}$).
- The n-dimensional *torus* $\mathbb{T}^n$ (the direct product of n copies of the circle).

FACT : **Any connected Abelian Lie group is isomorphic to**

$$\mathbb{T}^k \times \mathbb{R}^\ell$$

for some integers $k, \ell \geq 0$**.**

(Note that

$$\mathbb{S}^1 \times \mathbb{R} \cong \mathbb{C}^\times = \mathsf{GL}\,(1, \mathbb{C}).)$$

FACT : **Any compact Abelian Lie group is (up to isomorphism) of the form**

$$\mathbb{T}^k \times \mathsf{F}$$

for some integer $k \geq 0$**, where** F **is a finite Abelian group.**

FACT : The ONLY connected Lie groups (up to isomorphism) of dimension 1 are $\mathbb{R}$ and $\mathbb{T}^1 = \mathbb{S}^1$.

FACT : The ONLY connected Lie groups (up to isomorphism) of dimension 2 are

– $\mathbb{T}^k \times \mathbb{R}^{2-k}, \quad k = 0, 1, 2$

– $\mathsf{GA}^+(1, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix} \mid \alpha, \beta \in \mathbb{R}, \alpha > 0 \right\}$.

- The *special linear group* $\mathsf{SL}(n, \mathbb{k})$:

$$\mathsf{SL}(n, \mathbb{k}) = \{g \in \mathsf{GL}(n, \mathbb{k}) \mid \det g = 1\}.$$

- The *orthogonal group* $\mathsf{O}(n)$:

$$\mathsf{O}(n) = \left\{g \in \mathsf{GL}(n, \mathbb{R}) \mid g^\top g = I\right\}.$$

- The *special orthogonal group* $\mathsf{SO}(n)$:

$$\mathsf{SO}(n) = \mathsf{SL}(n, \mathbb{R}) \cap \mathsf{O}(n).$$

- The *unitary group* $\mathsf{U}(n)$:

$$\mathsf{U}(n) = \{g \in \mathsf{GL}(n, \mathbb{C}) \mid g^* g = I\}.$$

- The *special unitary group* $\mathsf{SU}(n)$:

$$\mathsf{SU}(n) = \mathsf{SL}(n, \mathbb{C}) \cap \mathsf{U}(n).$$

- The *complex orthogonal group* $\mathsf{O}(n, \mathbb{C})$:

$$\mathsf{O}(n, \mathbb{C}) = \left\{ g \in \mathsf{GL}(n, \mathbb{C}) \mid g^\top g = I \right\}.$$

- The *complex special orthogonal group* $\mathsf{SO}(n, \mathbb{C})$:

$$\mathsf{SO}(n, \mathbb{C}) = \mathsf{SL}(n, \mathbb{C}) \cap \mathsf{O}(n, \mathbb{C}).$$

- The *pseudo-orthogonal group* $\mathsf{O}(p, q)$:

$$\mathsf{O}(p, q) = \left\{ g \in \mathsf{GL}(n, \mathbb{R}) \mid g^\top I_{p,q}\, g = I_{p,q} \right\},$$

where $n = p + q$ and $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$.

- The *special pseudo-orthogonal group* $\mathsf{SO}(p, q)$:

$$\mathsf{SO}(p, q) = \mathsf{SL}(n, \mathbb{R}) \cap \mathsf{O}(p, q).$$

- The *symplectic group* $\mathsf{Sp}(n, \Bbbk)$:

  $\mathsf{Sp}(n, \Bbbk) = \left\{ g \in \mathsf{GL}(2n, \Bbbk) \mid g^\top J_{n,n}\, g = J_{n,n} \right\}$,

  where $J_{n,n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

  (For the members of $\mathsf{Sp}(n, \Bbbk)$ - the symplectic matrices over $\Bbbk$, the determinant is automatically 1.)

- The *compact symplectic group* $\mathsf{Sp}(n)$:

  $$\mathsf{Sp}(n) = \mathsf{Sp}(n, \mathbb{C}) \cap \mathsf{U}(2n).$$

  (The group $\mathsf{Sp}(n)$ is also known as the *unitary group over the quaternions*:

  $$\mathsf{Sp}(n) \cong \{ g \in \mathsf{GL}(n, \mathbb{H}) \mid g^* g = I \} ;$$

  again, for the elements of $\mathsf{Sp}(n)$, the determinant is automatically 1.)

- The *pseudo-unitary group* $\mathsf{U}(p,q)$:

  $$\mathsf{U}(p,q) = \{g \in \mathsf{GL}(n,\mathbb{C}) \mid g^* I_{p,q}\, g = I_{p,q}\},$$

  where $n = p + q$ and $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$.

- The *special pseudo-unitary group* $\mathsf{SU}(p,q)$:

  $$\mathsf{SU}(p,q) = \mathsf{SL}(n,\mathbb{C}) \cap \mathsf{U}(p,q).$$

- The *pseudo-symplectic group* $\mathsf{Sp}(p,q)$:

  $$\mathsf{Sp}(p,q) = \mathsf{Sp}(n,\mathbb{C}) \cap \mathsf{U}(2p,2q),$$

  where $n = p + q$.

  (The group $\mathsf{Sp}(p,q)$ is also known as the *pseudo-unitary group over the quaternions*:

  $$\mathsf{Sp}(p,q) \cong \{g \in \mathsf{GL}(n,\mathbb{H}) \mid g^* I_{p,q}\, g = I_{p,q}\}.)$$

- The group $\mathsf{O}^*(2n)$:

$$\mathsf{O}^*(2n) = \left\{ g \in \mathsf{U}(n,n) \mid g^\top I_{n,n} J_{n,n} g = I_{n,n} J_{n,n} \right\},$$

where $I_{n,n} J_{n,n} = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$.

- The group $\mathsf{SO}^*(2n)$:

$$\mathsf{SO}^*(2n) = \mathsf{SL}(2n, \mathbb{C}) \cap \mathsf{O}^*(2n).$$

- The group $\mathsf{SU}^*(2n)$:

$$\mathsf{SU}^*(2n) = \left\{ \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} \in \mathsf{SL}(2n, \mathbb{C}) \mid z_1, z_2 \in \mathbb{C}^{n \times n} \right\}.$$

(The group $\mathsf{SU}^*(2n)$ is the *special linear group over quaternions*, in its complex guise :

$$\mathsf{SU}^*(2n) \cong \mathsf{SL}(n, \mathbb{H}).)$$

- The *triangular group* $\mathsf{T}(n,\Bbbk)$:

  $$\mathsf{T}(n,\Bbbk) = \left\{ g \in \mathsf{GL}(n,\Bbbk) \mid g_{ij} = 0 \text{ for } i > j \right\}.$$

  (The members of $\mathsf{T}(n,\Bbbk)$ are invertible upper-triangular matrices over $\Bbbk$. This is an example of a **solvable Lie group**.)

- The *unipotent group* $\mathsf{T}^u(n,\Bbbk)$:

  $$\mathsf{T}^u(n,\Bbbk) = \left\{ g \in \mathsf{GL}(n,\Bbbk) \mid g_{ij} = \delta_{ij} \text{ for } i \geq j \right\},$$

  where δ_{ij} denotes the *Kronecker symbol*.

  (The members of $\mathsf{T}^u(n,\Bbbk)$ are invertible upper-triangular matrices over $\Bbbk$ with 1's on the main diagonal. This is an example of a **nilpotent Lie group**.)

  Note that

  - $\mathsf{T}^u(2,\Bbbk) \cong \Bbbk$;
  - $\mathsf{T}^u(3,\mathbb{R}) = \mathsf{Heis}$ (the *Heisenberg group*).

- The *Euclidean group* $\mathsf{E}(n)$:

$$\mathsf{E}(n) = \left\{ \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \mid A \in \mathsf{O}(n),\ c \in \mathbb{R}^n \right\}.$$

- The *special Euclidean group* $\mathsf{SE}(n)$:

$$\mathsf{SE}(n) = \left\{ \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \mid A \in \mathsf{SO}(n),\ c \in \mathbb{R}^n \right\}.$$

- The *symmetric group* $\mathfrak{S}_n$:

$$\mathfrak{S}_n = \{\sigma \mid \sigma \text{ is a permutation on } n \text{ elements}\}.$$

  – Any *finite group* is a matrix Lie group (of dimension 0).

FACT : **Any compact Lie group is (isomorphic to) a matrix Lie group**. (More precisely, any compact Lie group is isomorphic to a closed subgroup of the orthogonal group $\mathsf{O}(m)$ or the unitary group $\mathsf{U}(n)$ for some positive integers m and n, respectively).

- The following matrix Lie groups

$$\mathrm{SL}\,(n,\mathbb{R}),\ \mathrm{SL}\,(n,\mathbb{C}),\ \mathrm{SU}^*\,(2n),\ \mathrm{SO}\,(p,q),$$
$$\mathrm{SO}\,(n,\mathbb{C}),\ \mathrm{Sp}\,(n,\mathbb{R}),\ \mathrm{Sp}\,(n,\mathbb{C}),\ \mathrm{SU}\,(p,q),$$
$$\mathrm{Sp}\,(p,q),\quad \text{and}\quad \mathrm{SO}^*\,(2n)$$

  are known as the (real) **classical groups**.

  - The *compact* classical groups are

  $$\mathrm{SO}\,(n),\ \mathrm{Sp}\,(n),\quad \text{and}\quad \mathrm{SU}\,(n).$$

  - The classical groups are ALL *connected*, except for $\mathrm{SO}\,(p,q)$ with $p,q > 0$, which has two connected components.

(Note that

$$\mathrm{SO}_0\,(1,n) = \mathrm{Lor}\,(n),\qquad n \geq 2$$

is the *Lorentz group* of order n.)

FACT : **All the classical groups, except for the special linear groups** (i.e., $\mathsf{SL}(n,\mathbb{R})$, $\mathsf{SL}(n,\mathbb{C})$, $\mathsf{SL}(n,\mathbb{H}) \cong \mathsf{SU}^*(2n)$), **can be viewed as automorphism groups of forms**; these are *groups of linear transformations* (over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) *that preserve a non-degenerate form* (which may be symmetric, skew-symmetric, Hermitian or skew-Hermitian) *and of determinant equal to* 1.

For example,

– the special orthogonal group $\mathsf{SO}(n)$ may be realized as the group of linear transformations (of determinant equal to 1) on $\mathbb{R}^n$ preserving the symmetric bilinear form (the standard Euclidean structure)

$$\phi(x,y) = x^\top y = x_1 y_1 + \cdots + x_n y_n.$$

(These transformations are exactly the *rotations* about the origin.)

Automorphism groups of forms

Group	**Field**	**Form**
$\mathsf{SO}\,(p,q)$	$\mathbb{R}$	symmetric
$\mathsf{SO}\,(n,\mathbb{C})$	$\mathbb{C}$	symmetric
$\mathsf{Sp}\,(n,\mathbb{R})$	$\mathbb{R}$	skew-symmetric
$\mathsf{Sp}\,(n,\mathbb{C})$	$\mathbb{C}$	skew-symmetric
$\mathsf{SU}\,(p,q)$	$\mathbb{C}$	Hermitian
$\mathsf{Sp}\,(p,q)$	$\mathbb{H}$	Hermitian
$\mathsf{SO}^*\,(2n)$	$\mathbb{H}$	skew-Hermitian

If ϕ is a non-degenerate (bilinear or sesquiliniar) form on the finite dimensional vector space E (over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), then its associated automorphism group is

$$\mathsf{Aut}\,(\phi) = \{a \in \mathsf{GL}\,(E) \mid \phi(ax, ay) = \phi(x, y)\}\,.$$

(Note that

$$\mathbb{H}^{n\times 1} = \mathbb{H}^n = \mathbb{C}^n \oplus \mathbf{j}\mathbb{C}^n$$

is regarded as a *right* vector space over $\mathbb{H}$.)

4. **Matrix Lie algebras.**

- DEFINITION : A **matrix Lie algebra** is any *Lie subalgebra* of (the matrix space)

$$\mathbb{k}^{n\times n} = \mathfrak{gl}\,(n, \mathbb{k}), \quad \mathbb{k} \in \{\mathbb{R}, \mathbb{C}\}$$

(for some positive integer n).

- REMARK : Clearly, any matrix Lie algebra is an abstract *Lie algebra*. Quite remarkably, the converse is also true. This is a hard and deep result (due to the Russian mathematician Igor ADO) :
Any finite-dimensional Lie algebra over $\mathbb{R}$ has a one-to-one representation on some finite-dimensional complex vector space
(i.e., it is isomorphic to a matrix Lie algebra).

- Let G be a matrix Lie group :

$$\mathsf{G} \leq \mathsf{GL}\,(n, \Bbbk) \subset \Bbbk^{n \times n}.$$

In particular, G is a smooth submanifold of the Euclidean space $\Bbbk^{n^2}$.

The *tangent space* to G at the identity $1 = I_n \in \mathsf{G}$

$$T_1\mathsf{G} = \{\dot{\alpha}(0) \,|\, \alpha : (-\varepsilon, \varepsilon) \to \mathsf{G},\ \alpha(0) = 1\}$$

is a matrix Lie algebra, called the **Lie algebra** of G.

- The Lie algebra of G, denoted by $\mathfrak{g}$, is isomorphic to each of the following (matrix) Lie algebras :

  - $L(\mathsf{G}) = \left\{X \in \Bbbk^{n \times n} \,|\, e^{tX} \in \mathsf{G},\ \forall t \in \mathbb{R}\right\}$.
  - $\mathfrak{X}^L(\mathsf{G}) = \{X_A \,|\, X_A : g \mapsto gA,\ A \in T_1\mathsf{G}\}$.
  - $\{\sigma \,|\, \sigma : \mathbb{R} \to \mathsf{G}$ is 1-parameter subgroup of $\mathsf{G}\}$.

- The restriction (to $\mathfrak{g}$) of the exponential map, $\exp_{\mathsf{G}} = \exp|_{\mathfrak{g}}$, carries (the Lie algebra) $\mathfrak{g}$ into G. (In general, this map is neither one-to-one nor onto.) We list some of the basic properties of the **exponential map**

$$\exp_{\mathsf{G}} : \mathfrak{g} \to \mathsf{G}, \quad A \mapsto e^{A}.$$

  - Any 1-parameter subgroup of G (i.e., a continuous group homomorphism $\sigma : \mathbb{R} \to \mathsf{G}$) has the form

$$t \mapsto e^{tA}, \quad A \in \mathfrak{g}.$$

  - The flow of a left-invariant vector field $X_A : g \mapsto gA$ is given by

$$\varphi_t(g) = g e^{tA}, \quad A \in \mathfrak{g}.$$

  - The derivative of the exponential map at 0 is given by $D\exp_{\mathsf{G}}(0) = \mathrm{id}_{\mathfrak{g}}$, and hence (by the Inverse Function Theorem) $\exp_{\mathsf{G}}$ is locally a diffeomeorphism at 0.

– For $A \in \mathfrak{g}$,

$$\exp(A) = \lim_{k\to\infty}\left(I_n + \frac{1}{k}A\right)^k.$$

– For $A, B \in \mathfrak{g}$,

$$\exp(A+B) = \lim_{k\to\infty}\left[\exp\left(\frac{1}{k}A\right)\exp\left(\frac{1}{k}B\right)\right]^k.$$

(This formula relates addition in $\mathfrak{g}$ to multiplication in G.)

– For $A, B \in \mathfrak{g}$,

$$\exp([A,B]) =$$

$$\lim_{k\to\infty}\left[\exp\left(\frac{1}{k}A\right)\exp\left(\frac{1}{k}B\right)\exp\left(-\frac{1}{k}A\right)\exp\left(-\frac{1}{k}B\right)\right]^{k^2}.$$

(This formula relates the Lie bracket/commutator in $\mathfrak{g}$ to the group commutator in G.)

EXAMPLES of (MATRIX) LIE ALGEBRAS

- The Lie algebra of *scalar matrices*

$$\mathfrak{s}(n, \mathbb{k}) = \{\lambda I_n \mid \lambda \in \mathbb{k}\}.$$

- The *Abelian Lie algebra* $\mathbb{k}^n$ (the vector space $\mathbb{k}^n$ equipped with the trivial Lie multiplication).

- FACT : Up to isomorphism, there is ONLY ONE real Lie algebra of dimension 1, namely (the Abelian Lie algebra) $\mathfrak{g} = \mathbb{R}$.

- FACT : There are ONLY TWO distinct real Lie algebras of dimension 2 : the Abelian Lie algebra $\mathbb{R}^2$ and $\mathfrak{r}_2$.

  ($\mathfrak{r}_2$ is isomorphic to the Lie algebra of the matrix Lie group $\mathsf{R}_2 = \mathsf{GA}^+(1, \mathbb{R})$:

$$\mathfrak{r}_2 \cong \left\{ \begin{bmatrix} 0 & 0 \\ \beta & \alpha \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.)$$

- The Lie algebra of the *special linear group*:

$$\mathfrak{sl}\,(n, \mathbb{k}) = \{X \in \mathfrak{gl}\,(n, \mathbb{k}) \,|\, \operatorname{tr} X = 0\}\,.$$

- The *orthogonal Lie algebra* $\mathfrak{so}\,(n)$:

$$\mathfrak{so}\,(n) = \left\{X \in \mathfrak{gl}\,(n, \mathbb{R}) \,|\, X^\top + X = 0\right\}.$$

- The *unitary Lie algebra* $\mathfrak{u}\,(n)$:

$$\mathfrak{u}\,(n) = \{X \in \mathfrak{gl}\,(n, \mathbb{C}) \,|\, X^* + X = 0\}\,.$$

- The *special unitary Lie algebra* $\mathfrak{su}\,(n)$:

$$\mathfrak{su}\,(n) = \{X \in \mathfrak{gl}\,(n, \mathbb{C}) \,|\, X^* + X = 0,\ \operatorname{tr} X = 0\}\,.$$

- The *complex orthogonal Lie algebra* $\mathfrak{so}(n, \mathbb{C})$:

$$\mathfrak{so}(n, \mathbb{C}) = \left\{ X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^\top + X = 0 \right\}.$$

- The *pseudo-orthogonal* Lie algebra $\mathfrak{so}(p, q)$:

$$\mathfrak{so}(p, q) = \left\{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid X^\top I_{p,q} + I_{p,q} X = 0 \right\},$$

where $n = p + q$ and $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$.

- The *symplectic Lie algebra* $\mathfrak{sp}(n, \Bbbk)$:

$$\mathfrak{sp}(n, \Bbbk) = \left\{ X \in \mathfrak{gl}(2n, \Bbbk) \mid X^\top J_{n,n} + J_{n,n} X = 0 \right\},$$

where $J_{n,n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

- The *pseudo-unitary Lie algebra* $\mathfrak{su}(p,q)$:

  $$\mathfrak{su}(p,q) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X^* I_{p,q} + I_{p,q} X = 0\},$$

  where $n = p + q$ and $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$.

- The *pseudo-symplectic Lie algebra* $\mathfrak{sp}(p,q)$:

  $$\mathfrak{sp}(p,q) = \{X \in \mathfrak{gl}(n,\mathbb{H}) \mid X^* I_{p,q} + I_{p,q} X = 0\},$$

  where $n = p + q$ and $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$.

  In particular, the Lie algebra (of the *compact symplectic group*) $\mathfrak{sp}(n)$ is isomorphic to

  $$\left\{ \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} \mid z_1^* + z_1 = 0,\ z_2^\top = z_2 \right\}.$$

  (In fact,

  $$\mathfrak{sp}(n) \cong \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{u}(2n).)$$

- The Lie algebra of the group $\mathsf{SO}^*(2n)$:

  $$\mathfrak{so}^*(2n) = \left\{ X \in \mathfrak{su}(n,n) \mid X^\top I_{n,n} J_{n,n} + I_{n,n} J_{n,n} X = 0 \right\},$$

  where $I_{n,n} J_{n,n} = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$.

- The Lie algebra of the group $\mathsf{SU}^*(2n)$:

  $$\begin{aligned} \mathfrak{su}^*(2n) &= \{ X \in \mathfrak{gl}(n, \mathbb{H}) \mid \mathrm{Re}(\mathrm{tr}\, X) = 0 \} \\ &= \left\{ X = \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} \mid z_1, z_2 \in \mathfrak{gl}(n, \mathbb{C}),\ \mathrm{tr}\, X = 0 \right\}. \end{aligned}$$

- The Lie algebra of the *triangular group* $\mathsf{T}(n, \Bbbk)$:

  $$\mathfrak{t}(n, \Bbbk) = \left\{ X \in \mathfrak{gl}(n, \Bbbk) \mid X_{ij} = 0 \text{ for } i > j \right\}.$$

  (The elements of $\mathfrak{t}(n, \Bbbk)$ are upper-triangular matrices over $\Bbbk$. This is an example of a **solvable Lie algebra**.)

- The Lie algebra of the *unipotent group* $\mathsf{T}^u(n, \Bbbk)$:

  $$\mathfrak{t}^u(n, \Bbbk) = \left\{ X \in \mathfrak{gl}(n, \Bbbk) \mid X_{ij} = \delta_{ij} \text{ for } i \geq j \right\}.$$

  (The elements of $\mathfrak{t}^u(n, \Bbbk)$ are *strictly* upper-triangular matrices over $\Bbbk$. This is an example of a **nilpotent Lie algebra**.)

  In particular, the Lie algebra (of strictly upper-triangular 3×3 matrices)

  $$\mathfrak{heis} = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

  is the *Heisenberg Lie algebra*.

- The *Euclidean Lie algebra* $\mathfrak{se}\,(n)$:

$$\mathfrak{se}\,(n) = \left\{ \begin{bmatrix} 0 & 0 \\ c & A \end{bmatrix} \mid A \in \mathfrak{so}\,(n),\ c \in \mathbb{R}^n \right\}.$$

In particular, the Lie algebra

$$\mathfrak{se}\,(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -\theta \\ y & \theta & 0 \end{bmatrix} \mid x, y, \theta \in \mathbb{R} \right\}$$

is isomorphic with the Lie algebra of the group of *translations* and *rotations* of (the Euclidean plane) $\mathbb{R}^2$.

- The Lie algebra

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & t & 0 \\ y & 0 & t \end{bmatrix} \mid x, y, t \in \mathbb{R} \right\}$$

is isomorphic with the Lie algebra of the group of *translations* and *dilations* of (the Euclidean plane) $\mathbb{R}^2$.

5. A glimpse at elementary Lie theory.

- LIE THEORY is a fundamental part of mathematics. It is a subject which permeates many branches of *modern mathematics* and *mathematical physics*.

- If we are willing to ignore a number of details, the **elementary Lie theory** has
  - TWO ingredients
  - THREE correspondences.

  (For each of the three correspondences, there is a *direct part* and an *inverse part*.)

- The two ingredients are *Lie groups* and *Lie algebras*.

- The **Lie algebra** of the Lie group G is the *tangent space* $T_e\mathsf{G}$ to G at the identity $e \in \mathsf{G}$. It is isomorphic to (and hence identified with) the Lie algebra $\mathfrak{X}^L(\mathsf{G})$ of *left-invariant vector fields* on G.

$$(X \in \mathfrak{X}^L(\mathsf{G}) \iff (L_g)_* X = X,\ g \in \mathsf{G}.)$$

- FACT : Any matrix Lie group is a Lie group

- FACT : Not every Lie group is (isomorphic to) a matrix Lie group, but **a Lie group is always locally isomorphic to a matrix Lie group**.

- FIRST CORRESPONDENCE (direct part):

  To each Lie group there corresponds a Lie algebra.

  The essential tool in studying the relationship between a Lie group and its Lie algebra is the *exponential map*.

  For $X \in \mathfrak{g} = \mathfrak{X}^L(\mathsf{G})$, let $\gamma_X : \mathbb{R} \to \mathsf{G}$ denote the *integral* curve of (the complete vector field) X with initial condition e. Then the map

  $$\exp_{\mathsf{G}} : \mathfrak{g} \to \mathsf{G}, \quad X \mapsto \gamma_X(1)$$

  is smooth and its tangent map (differential) at 0

  $$(\exp_{\mathsf{G}})_{*,0} : T_0\,\mathfrak{g} = \mathfrak{g} \to T_e\mathsf{G} = \mathfrak{g}$$

  is the identity.

- FIRST CORRESPONDENCE (the inverse part) :

  Two connected Lie groups with isomorphic Lie algebras are not necessarily isomorphic, but they must have covering groups that are isomorphic.

  If the connected Lie groups G_1, G_2 with $\mathfrak{g}_1 \cong \mathfrak{g}_2$ are *simply connected* (i.e., their fundamental groups are trivial), then $G_1 \cong G_2$.

  FACT : If $\widetilde{G}$ is the *universal covering group* of the (connected) Lie group G, then any Lie group locally isomorphic to G is (isomorphic to) a quotient of $\widetilde{G}$ by a discrete subgroup of G which lies in the centre of $\widetilde{G}$.

  (NB : The universal cover of a matrix Lie group may not be a matrix Lie group.)

- SECOND CORRESPONDENCE (direct part) :

  To each subgroup of a certain kind corresponds a Lie subalgebra.

  If H is a Lie subgroup of the Lie group G, then the exponential map of H is the restriction to $\mathfrak{h}$ of the exponential map of G, and

$$\mathfrak{h} = \{X \in \mathfrak{g} \,|\, \exp(tX) \in \mathsf{H} \text{ for all } t \in \mathbb{R}\}\,.$$

- SECOND CORRESPONDENCE (the inverse part) :

  The correspondence of analytic subgroups to subalgebras of Lie algebras is one-to-one and onto.

  Given a Lie group G with Lie algebra $\mathfrak{g}$, if $\mathfrak{h}$ is any Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$.

- THIRD CORRESPONDENCE (direct part):

  To each smooth homomorphism of Lie groups there corresponds a homomorphism of Lie algebras.

  If $\phi : \mathsf{G} \to \mathsf{H}$ is a smooth homomorphism of Lie groups, then the (tangent) map

$$d\phi = \phi_* : \mathfrak{g} \to \mathfrak{h}$$

  satisfies

$$\exp_{\mathsf{H}}(d\phi(X)) = \phi(\exp_{\mathsf{G}}(X))$$

  for all $X \in \mathfrak{g}$. In other words, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\phi} & \mathfrak{h} \\ {\scriptstyle \exp_{\mathsf{G}}}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{\mathsf{H}}} \\ \mathsf{G} & \xrightarrow{\phi} & \mathsf{H} \end{array}$$

  commutes. Moreover, for all $X, Y \in \mathfrak{h}$,

$$d\phi([X,Y]_{\mathsf{G}}) = [d\phi(X), d\phi(Y)]_{\mathsf{H}}.$$

- THIRD CORRESPONDENCE (the inverse part) :

  The correspondence of homomorphisms of Lie groups to homomorphisms of Lie algebras is one-to-one if the group is connected.

  If ϕ_1 and ϕ_2 are smooth homomorphisms between Lie groups G and H such that $d\phi_1 = d\phi_2$, then $\phi_1 = \phi_2$ on G^0 (the connected component of the identity).

  Moreover, if G an H are connected Lie groups with G simply connected, for any Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$, there is a unique smooth homomorphism of Lie groups $\phi : G \to H$ such that $d\phi = \Phi$.

- REMARK : The **Lie functor** taking a (real, finite-dimensional) Lie group G to its associated Lie algebra $\mathfrak{g}$ and a smooth homomorphism ϕ of Lie grups to its associated Lie map $d\phi$ is an *equivalence of categories* between the category of connected and simply connected Lie groups and the category of Lie algebras.

  The study of Lie groups can be reduced to questions in (the vastly simpler realm of) linear algebra.

6. **Life beyond elementary Lie theory.**

The ground field is $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$.

- There is a *rough classification* of Lie algebras over $\mathbb{k}$, which reflects the degree to which a given Lie algebra fails to be *Abelian*.

  A Lie algebra $\mathfrak{g}$ is said to be **Abelian** if all Lie brackets are zero.

- FACT : For a connected Lie group G with Lie algebra $\mathfrak{g}$, G is Abelian if and only if $\mathfrak{g}$ is Abelian.

- A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an **ideal** if it satisfies the condition

$$[X, Y] \in \mathfrak{h} \quad \text{for all } X \in \mathfrak{h},\, Y \in \mathfrak{g}.$$

  (Just as connected subgroups of Lie groups correspond to subalgebras of its Lie algebra, the notion of *ideal* in a Lie algebra corresponds to the notion of *normal subgroup*.)

- A non-Abelian Lie algebra $\mathfrak{g}$ is called **simple** if it has no nontrivial ideals.

  A **simple Lie group** is a connected Lie group with a simple Lie algebra.

- FACT : A connected Lie group G is simple if and only if it has no proper normal Lie subgroups.

- For a Lie algebra $\mathfrak{g}$ we define
  - the **lower central series** (of ideals)

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots ,$$

    where $\mathfrak{g}_{j+1} = \left[\mathfrak{g}, \mathfrak{g}_j\right], \;\; j = 0, 1, \ldots$

  - **commutator series** (of ideals)

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots ,$$

    where $\mathfrak{g}^{j+1} = \left[\mathfrak{g}^j, \mathfrak{g}^j\right], \;\; j = 0, 1, \ldots$

- A Lie algebra $\mathfrak{g}$ is called
  - **nilpotent** if $\mathfrak{g}_j = 0$ for some j.
  - **solvable** if $\mathfrak{g}^j = 0$ for some j.
  - **semisimple** if it has no nontrivial solvable ideals.

A connected Lie group is **nilpotent**, **solvable** or **semisimple** if its Lie algebra is nilpotent, solvable or semisimple, respectively.

- FACT : Every Abelian Lie group G (respectively Abelian Lie algebra $\mathfrak{g}$) is nilpotent.

- FACT : Every nilpotent Lie group G (respectively nilpotent Lie algebra $\mathfrak{g}$) is solvable.

- FACT : Every simple Lie group G (respectively simple Lie algebra $\mathfrak{g}$) is semisimple.

- Since the sum of two solvable ideals of a Lie algebra is a solvable ideal, every Lie algebra $\mathfrak{g}$ has a maximal solvable ideal, called the **radical** of $\mathfrak{g}$ and denoted by $\mathrm{rad}\,(\mathfrak{g})$.

- FACT : For any Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{rad}\,(\mathfrak{g})$ is semisimple.

  Any Lie algebra $\mathfrak{g}$ thus fits into an exact sequence

$$0 \longrightarrow \mathrm{rad}\,(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathrm{rad}\,(\mathfrak{g}) \longrightarrow 0,$$

  where the first Lie algebra is *solvable* and the last is *semisimple*.

- LEVI DECOMPOSITION : **Any** (finite-dimensional real) **Lie algebra $\mathfrak{g}$ is the semidirect sum of its** (solvable) **radical** rad $(\mathfrak{g})$ **and a semisimple subalgebra** $\mathfrak{l}$:

$$\mathfrak{g} = \text{rad}\,(\mathfrak{g}) \leftthreetimes \mathfrak{l}.$$

(Eugenio LEVI (1883-1917).)

To study Lie groups/algebras we need to understand individually the theories of *solvable* and *semisimple* Lie algebras.

Of these, the former is relatively easy, whereas the latter is extraordinarily rich.

- FACT : A Lie algebra $\mathfrak{g}$ is *semisimple* if and only if

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m$$

with $\mathfrak{g}_j$ ideals that are each *simple* Lie algebras.

- In his 1894 thesis, Elie CARTAN (1869-1951), correcting and improving earlier work of Wilhelm KILLING (1847-1923), classified the *simple Lie algebras* over $\mathbb{C}$.

  Any simple Lie algebra over $\mathbb{C}$ is isomorphic to exactly one of the following :

  (a) a Lie algebra of *classical type*

$$(\mathbf{A}_\ell): \qquad \mathfrak{a}_\ell = \mathfrak{sl}\,(\ell+1,\mathbb{C}), \ \ \ell \geq 1.$$

$$(\mathbf{B}_\ell): \qquad \mathfrak{b}_\ell = \mathfrak{so}\,(2\ell+1,\mathbb{C}), \ \ \ell \geq 2.$$

$$(\mathbf{C}_\ell): \qquad \mathfrak{c}_\ell = \mathfrak{sp}\,(\ell,\mathbb{C}), \ \ \ell \geq 3.$$

$$(\mathbf{D}_\ell): \qquad \mathfrak{d}_\ell = \mathfrak{so}\,(2\ell,\mathbb{C}), \ \ \ell \geq 4.$$

  (b) an *exceptional* Lie algebra

$$\mathfrak{e}_6, \ \mathfrak{e}_7, \ \mathfrak{e}_8, \ \mathfrak{f}_4, \ \mathfrak{g}_2.$$

CLASSIFICATION of SIMPLE REAL LIE ALGEBRAS

Up to isomorphism every simple real Lie algebra is in the following list, and everything in the list is a simple real Lie algebra :

(a) the Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is a *complex simple Lie algebra* :

$$\mathfrak{a}_\ell\ (\ell \geq 1),\ \mathfrak{b}_\ell\ (\ell \geq 2),\ \mathfrak{c}_\ell\ (\ell \geq 3),\ \mathfrak{d}_\ell\ (\ell \geq 4)$$

$$\mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8,\ \mathfrak{f}_4,\ \mathfrak{g}_2.$$

(b) the *compact real form* of any $\mathfrak{g}$ as in (a) :

$$\mathfrak{su}\,(\ell+1),\ \ \ell \geq 1.$$

$$\mathfrak{so}\,(2\ell+1),\ \ \ell \geq 2.$$

$$\mathfrak{sp}\,(\ell),\ \ \ell \geq 3.$$

$$\mathfrak{so}\,(2\ell),\ \ \ell \geq 4.$$

(c) the *classical matrix Lie algebras*

$$\mathfrak{su}\,(p,q),\;\; p \geq q > 0,\; p+q \geq 2.$$

$\mathfrak{so}\,(p,q),\;\; p > q > 0,\, p{+}q$ odd, $p{+}q \geq 5$ or $p \geq q > 0, p+q$ even, $p+q \geq 8$.

$$\mathfrak{sp}\,(p,q),\;\; p \geq q > 0,\; p+q \geq 3.$$

$$\mathfrak{sp}\,(n),\;\; n \geq 3.$$

$$\mathfrak{so}^*\,(2n),\;\; n \geq 4.$$

$$\mathfrak{sl}\,(n,\mathbb{R}),\;\; n \geq 3.$$

$$\mathfrak{sl}\,(n,\mathbb{H}) = \mathfrak{su}^*\,(2n),\;\; n \geq 2.$$

(d) the 12 *exceptional* noncomplex noncompact simple Lie algebras ...

NB : The ONLY isomorphism among Lie algebras in the above list is

$$\mathfrak{so}^*\,(8) \cong \mathfrak{so}\,(6,2).$$

The COMPACT Simple Lie Groups

The collection

$$\mathsf{A}_\ell\ (\ell \geq 1),\ \mathsf{B}_\ell\ (\ell \geq 2),\ \mathsf{C}_\ell\ (\ell \geq 3),\ \mathsf{D}_\ell\ (\ell \geq 4)$$
$$\mathsf{E}_2,\ \mathsf{E}_7,\ \mathsf{E}_8,\ \mathsf{F}_4 \quad \text{and} \quad \mathsf{G}_2.$$

is precisely the set, without repetion, of all the local isomorphism classes of the compact simple Lie groups.

(NB : A_ℓ denotes the local isomophism class of $\mathsf{SU}\,(\ell+1)$, etc.)

Some redundancies occur in the lower values of the *rank* :

$$\begin{aligned} \mathsf{A}_1 &= \mathsf{B}_1 = \mathsf{C}_1 \\ \mathsf{D}_2 &= \mathsf{A}_1 \times \mathsf{A}_1 \\ \mathsf{B}_2 &= \mathsf{C}_2 \\ \mathsf{A}_3 &= \mathsf{D}_3 \\ \mathsf{D}_1 &\ \text{is}\ \ \text{Abelian.} \end{aligned}$$

(The *rank* of the Lie group G is the dimension of the maximal torus in G.)

- The *simple Lie groups*, and groups closely related to them, include
  - (many of) the *classical groups* of geometry. (These groups lie behind projective geometry and other geometries derived from it; they can be uniformly described as groups of isometries.)
  - some **exceptional groups**. (These groups are related to the isometries of projective planes over *octonion* algebras; they have dimension 78, 133, 248, 52 and 14, respectively.)

- FACT : The exceptional simple Lie group G_2 (of dimension 14) is (isomorphic to) the *automorphism group* of the (Cayley algebra) of *octonions*.

The REAL Classical Groups

Real matrices	**Complex matrices**	**Quaternionic matrices**
$\mathsf{SL}\,(n,\mathbb{R})$	$\mathsf{SL}\,(n,\mathbb{C})$	$\mathsf{SL}\,(n,\mathbb{H})$
$\mathsf{SO}\,(p,q)$	$\mathsf{SO}\,(n,\mathbb{C})$	$\mathsf{SO}^*\,(2n)$
$\mathsf{Sp}\,(n,\mathbb{R})$	$\mathsf{Sp}\,(n,\mathbb{C})$	$\mathsf{Sp}\,(p,q)$
	$\mathsf{SU}\,(p,q)$	$\mathsf{SU}^*\,(2n)$

Simple Lie Groups of Small Dimension

3	$\mathsf{SU}\,(2) = \mathsf{Sp}\,(1)$	$\mathsf{SO}\,(3)$	
3	$\mathsf{SL}\,(2,\mathbb{R}) = \mathsf{Sp}\,(1,\mathbb{R})$	$\mathsf{SO}_0\,(1,2)$	
6	$\mathsf{SL}\,(2,\mathbb{C}) = \mathsf{Sp}\,(1,\mathbb{C})$	$\mathsf{SO}_0\,(1,3)$	$\mathsf{SO}\,(3,\mathbb{C})$
8		$\mathsf{SL}\,(3,\mathbb{R})$	
8		$\mathsf{SU}\,(3)$	
8		$\mathsf{SU}\,(1,2)$	
10	$\mathsf{Sp}\,(2)$	$\mathsf{SO}\,(5)$	
10	$\mathsf{Sp}\,(1,1)$	$\mathsf{SO}_0\,(1,4)$	
10	$\mathsf{Sp}\,(2,\mathbb{R})$	$\mathsf{SO}_0\,(2,3)$	

NB : The groups on a given line all have the same Lie algebra.

- The three introductory examples of matrix Lie groups are as follows :
  - The connected component of the identity of E (2) (i.e., the *special Euclidean group* SE (2)) is *solvable*.
  - The rotation group SO (2) is *Abelian*.
  - The special unitary group SU (2) is *simple*.