Control and Integrability on $SO(3)$

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Outline

- Introduction
- Optimal control (on matrix Lie groups)
- An invariant control problem
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Introduction

A wide range of dynamical systems from

- classical mechanics
- quantum mechanics
- elasticity
- electrical networks
- molecular chemistry

can be modelled by invariant systems on matrix Lie groups.

Invariant control systems were first considered by Brockett (1972) and by Jurdjevic and Sussmann (1972).

Optimal control : invariant control systems and controls

Invariant control system

A left-invariant control system (evolving on some matrix Lie group G) is described by

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

The parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is a (smooth) embedding.

Admissible control

An admissible control is a map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".)

Optimal control : trajectories

Trajectory

A trajectory for an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$$

for almost every $t \in [0, T]$.

Controlled trajectory

A controlled trajectory is a pair $(g(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $g(\cdot)$ is the trajectory corresponding to $u(\cdot)$.

Optimal control : invariant control problems

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the (controlled) trajectories of a given left-invariant control system, subject to appropriate boundary conditions :

Left-invariant control problem (LiCP)

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

$$g(0) = g_0,\ g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$

$$\mathcal{J} = \frac{1}{2}\int_0^T L(u(t))\,dt \to \min.$$

The terminal time $T > 0$ can be either fixed or it can be free.

Optimal control : the left-invariant realization of $T^*\mathrm{G}$

Cotangent bundle

The cotangent bundle $T^*\mathrm{G}$ can be trivialized (from the left) such that

$$T^*\mathrm{G} = \mathrm{G} \times \mathfrak{g}^*.$$

Explicitly, $\xi \in T_g^*\mathrm{G}$ is identified with $(g, p) \in \mathrm{G} \times \mathfrak{g}^*$ via $p = dL_g^*(\xi)$:

$$\xi(gA) = p(A), \quad g \in \mathrm{G}, A \in \mathfrak{g}.$$

Each element (matrix) $A \in \mathfrak{g}$ defines a (smooth) function H_A on $T^*\mathrm{G}$:

$$H_A(\xi) = \xi(gA), \quad \xi \in T_g^*\mathrm{G}.$$

H_A is left-invariant (as a function on $\mathrm{G} \times \mathfrak{g}^*$), which is equivalent to saying that H_A is a function on $\mathfrak{g}^*$.

Optimal control : the symplectic structure of T^*G

Hamiltonian vector field

The canonical symplectic form ω on T^*G sets up a correspondence between (smooth) functions H on T^*G and vector fields $\vec{H}$ on T^*G :

$$\omega_\xi\left(\vec{H}(\xi), V\right) = dH(\xi)\cdot V, \quad V \in T_\xi(T^*G).$$

Each left-invariant Hamiltonian on T^*G is identified with its reduction on the dual space $\mathfrak{g}^*$.

Hamilton's equations

The equations of motion for the left-invariant Hamiltonian H are

$$\dot{g} = g\, dH(p) \quad \text{and} \quad \dot{p} = \mathrm{ad}^*_{dH(p)} p.$$

Optimal control : the Lie-Poisson bracket on $\mathfrak{g}^*$

The minus Lie-Poisson structure

The dual space $\mathfrak{g}^*$ has a natural Poisson structure :

$$\{F, G\}_{-}(p) = -p\left([dF(p), dG(p)]\right), \quad p \in \mathfrak{g}^*, \; F, G \in C^\infty(\mathfrak{g}^*).$$

If $(E_k)_{1 \leq k \leq m}$ is a basis for $\mathfrak{g}$ and

$$[E_i, E_j] = \sum_{k=1}^{m} c_{ij}^k \, E_k$$

then

$$\{F, G\}_{-}(p) = -\sum_{i,j,k=1}^{m} c_{ij}^k \, p_k \, \frac{\partial F}{\partial p_i} \, \frac{\partial G}{\partial p_j}.$$

Optimal control : the Maximum Principle

To a LiCP (with fixed terminal time) we associate

$$H_u^\lambda(\xi) = \lambda\, L(u) + p\,(\Xi(\mathbf{1}, u))\,, \quad \xi = (g, p) \in T^*\mathsf{G}.$$

Theorem (Pontryagin's Maximum Principle)

Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ is a solution for the LiCP. Then, there exists a curve $\xi(\cdot)$ with $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$ and $\lambda \leq 0$ such that

$$(\lambda, \xi(t)) \not\equiv (0,0) \quad (\textit{nontriviality})$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \quad (\textit{Hamiltonian system})$$

$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H_u^\lambda(\xi(t)) = \textit{constant}. \quad (\textit{maximization})$$

An extremal curve is called normal if $\lambda = -1$ (and abnormal if $\lambda = 0$).

Optimal control : control affine dynamics

Theorem (Krishnaprasad, 1993)

For the LiCP (with quadratic cost)

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

$$g(0) = g_0,\ g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + \cdots + c_\ell u_\ell^2(t)\right)\,dt \to \min \quad (T \textit{ is fixed})$$

every normal extremal is given by

$$\bar{u}_i(t) = \frac{1}{c_i} p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the vector field* $\vec{H}$ *corresponding to* $H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1} p(B_1)^2 + \cdots + \frac{1}{c_\ell} p(B_\ell)^2\right)$.

The rotation group $\mathsf{SO}(3) = \left\{a \in \mathsf{GL}(3,\mathbb{R}) : a^\top a = \mathbf{1},\ \det a = 1\right\}$ is a 3D compact connected matrix Lie group with associated Lie algebra

$$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

The standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We identify $\mathfrak{so}(3)$ with (the cross-product Lie algebra) $\mathbb{R}^3_\wedge$.

An invariant control problem : the single-input case

A left-invariant control problem on SO (3)

We consider the LiCP

$$\dot{g} = g\,(E_3 + uE_1)\,, \quad g \in \mathrm{SO}\,(3),\ u \in \mathbb{R}$$
$$g(0) = g_0, \quad g(T) = g_1 \ \ (g_0, g_1 \in \mathrm{SO}\,(3))$$
$$\mathcal{J} = \frac{1}{2}\int_0^T u^2(t)\,dt \to \min.$$

This problem models a variation of the classical elastic problem of Euler and Kirchhoff.

An invariant control problem : the Lie-Poisson bracket

$\mathfrak{so}(3)^*$ is identified with $\mathfrak{so}(3)$ via $\langle A, B \rangle = -\frac{1}{2}\operatorname{tr}(AB)$.

Each extremal curve $p(\cdot)$ is identified with a curve $P(\cdot)$ in $\mathfrak{so}(3)$ via

$$\langle P(t), A \rangle = p(t)(A), \quad A \in \mathfrak{so}(3).$$

The Lie-Poisson bracket on $\mathfrak{so}(3)^*$ is given by

$$\begin{aligned} \{F, G\}_{-}(p) &= -\sum_{i,j,k=1}^{3} c_{ij}^{k} p_k \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial p_j} \\ &= -\widehat{P} \bullet (\nabla F \times \nabla G) \end{aligned}$$

($p \in \mathfrak{so}(3)^*$ is identified with the vector $\widehat{P} = (P_1, P_2, P_3) \in \mathbb{R}^3$).

An invariant control problem : the equations of motion

The equation of motion becomes

$$\begin{aligned} \dot{F} &= \{F, H\}_- \\ &= \nabla F \bullet \left(\widehat{P} \times \nabla H\right). \end{aligned}$$

The scalar equations of motion

$$\begin{aligned} \dot{P}_1 &= \frac{\partial H}{\partial p_3} P_2 - \frac{\partial H}{\partial p_2} P_3 \\ \dot{P}_2 &= \frac{\partial H}{\partial p_1} P_3 - \frac{\partial H}{\partial p_3} P_1 \\ \dot{P}_3 &= \frac{\partial H}{\partial p_2} P_1 - \frac{\partial H}{\partial p_1} P_2. \end{aligned}$$

An invariant control problem : the extremal trajectories

Proposition

Given the LiCS, the extremal control is $\bar{u} = P_1$, *where* $P_1 : [0, T] \to \mathbb{R}$ *(together with* P_2 *and* P_3*) is a solution of the system of ODEs*

$$\begin{aligned} \dot{P}_1 &= P_2 \\ \dot{P}_2 &= P_1 P_3 - P_1 \\ \dot{P}_3 &= -P_1 P_2. \end{aligned}$$

The extremal trajectories are the intersections of

- the parabolic cylinders $P_1^2 + 2P_3 = 2H$
- the spheres $P_1^2 + P_2^2 + P_3^2 = C$.

Explicit integration : elliptic functions

Jacobi elliptic functions

The Jacobi elliptic functions $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$ and $\mathrm{dn}(\cdot, k)$ can be defined as the solution of the system of ODEs

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= -zx \\ \dot{z} &= -k^2 xy \end{aligned}$$

that satisfy the initial conditions : $x(0) = 0$, $y(0) = 1$, $z(0) = 1$.

Nine other elliptic functions are defined by taking reciprocals and quotients. In particular, we get

$$\mathrm{ns}(\cdot, k) = \frac{1}{\mathrm{sn}(\cdot, k)} \quad \text{and} \quad \mathrm{dc}(\cdot, k) = \frac{\mathrm{dn}(\cdot, k)}{\mathrm{cn}(\cdot, k)}.$$

Explicit integration : elliptic integrals

An elliptic integral is an integral of the type $\int R(x,y)\,dx$, where y^2 is a cubic or quartic polynomial in x and $R(\cdot,\cdot)$ denotes a rational function.

Simple elliptic integrals can be expressed in terms of inverses of appropriate Jacobi elliptic functions. Specifically (for $b < a \leq x$) :

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{dc}^{-1}\left(\frac{x}{a},\frac{b}{a}\right)$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{ns}^{-1}\left(\frac{x}{a},\frac{b}{a}\right).$$

Explicit integration : statement

Proposition

The reduced Hamilton equations

$$\dot{P}_1 = P_2, \quad \dot{P}_2 = P_1 P_3 - P_1, \quad \dot{P}_3 = -P_1 P_2$$

can be explicitly integrated by Jacobi elliptic functions (for $H^2 - C > 0$*) :*

$$\begin{aligned} P_1 &= \pm\sqrt{2(H - P_3)} \\ P_2 &= \pm\sqrt{C - 2(H - P_3) - P_3^2} \\ P_3 &= \frac{\alpha - \beta\delta\,\Phi((\alpha - \beta)M\delta t, \epsilon/\delta)}{1 - \delta\,\Phi((\alpha - \beta)M\delta t, \epsilon/\delta)}. \end{aligned}$$

$\alpha = H + \sqrt{H^2 - C}$, $\beta = H - \sqrt{H^2 - C}$, $M = \frac{H - \sqrt{H^2 - C} - 1}{4(H^2 - C)}$, $\epsilon^2 = 1$,
$\delta^2 = \frac{1 - \sqrt{H^2 - C} - H}{1 + \sqrt{H^2 - C} - H}$, and $\Phi(\cdot, k)$ is either $\mathrm{dc}(\cdot, k)$ or $\mathrm{ns}(\cdot, k)$.

Final remark

Invariant optimal control problems on matrix Lie groups other than the rotation group $\mathsf{SO}(3)$ (like

- the Euclidean groups $\mathsf{SE}(2)$ and $\mathsf{SE}(3)$
- the Lorentz groups $\mathsf{SO}_0(1,2)$ and $\mathsf{SO}_0(1,3)$
- the Heisenberg groups $\mathsf{H}(1)$ and $\mathsf{H}(2)$)

can also be considered.

It is to be expected that explicit integration (of the reduced Hamilton equations) will be possible in all these cases.