Integrability and Optimal Control

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Outline

- Introduction
- Preliminaries
- Integrability
- A class of optimal control problems
- An optimal control problem on the rotation group $\mathsf{SO}(3)$
- Final remark

Dynamical and control systems

A wide range of dynamical systems from

- classical mechanics
- quantum mechanics
- elasticity
- electrical networks
- molecular chemistry

can be modelled by invariant systems on matrix Lie groups.

Applied nonlinear control

Invariant control systems with control affine dynamics (evolving on matrix Lie groups of low dimension) arise in problems like

- the airplane landing problem
- the attitude problem (in spacecraft dynamics)
- the motion planning for wheeled robots
- the control of underactuated underwater vehicles
- the control of quantum systems
- the dynamic formation of the DNA

Invariant control systems

Invariant control systems were first considered by Brockett (1972) and by Jurdjevic and Sussmann (1972).

A left-invariant control system (evolving on some matrix Lie group G) is described by

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\ u \in \mathbb{R}^{\ell}.$$

The parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^{\ell} \to \mathfrak{g}$ is a (smooth) embedding.

Invariant control systems

Admissible control

An admissible control is a map $u(\cdot) : [O, T] \to \mathbb{R}^\ell$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".)

Trajectory

A trajectory for an admissible control $u(\cdot) : [O, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g : [O, T] \to \mathrm{G}$ such that

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$$

for almost every $t \in [O, T]$.

Invariant control systems

Controlled trajectory

A controlled trajectory is a pair $(g(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $g(\cdot)$ is the trajectory corresponding to $u(\cdot)$.

Control affine dynamics

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls :

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Optimal control problems

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the (controlled) trajectories of a given left-invariant control system, subject to appropriate boundary conditions :

Left-invariant control problem (LiCP)

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

$$g(0) = g_0,\ \ g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$

$$\mathcal{J} = \frac{1}{2}\int_0^T L(u(t))\,dt \to \min.$$

The terminal time $T > 0$ can be either fixed or it can be free.

Optimal control problems

The left-invariant realization of $T^*\mathsf{G}$

The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that

$$T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*.$$

Explicitly, $\xi \in T_g^*\mathsf{G}$ is identified with $(g, p) \in \mathsf{G} \times \mathfrak{g}^*$ via $p = dL_g^*(\xi)$:

$$\xi(gA) = p(A), \quad g \in \mathsf{G},\ A \in \mathfrak{g}.$$

(dL_g^* denotes the dual of the tangent map $dL_g = (L_g)_{*,\mathbf{1}} : \mathfrak{g} \to T_g\mathsf{G}$.)

Optimal control problems

Hamiltonian vector field

The canonical symplectic form ω on $T^*\mathrm{G}$ sets up a correspondence between (smooth) functions H on $T^*\mathrm{G}$ and vector fields $\vec{H}$ on $T^*\mathrm{G}$:

$$\omega_\xi\left(\vec{H}(\xi), V\right) = dH(\xi)\cdot V, \quad V \in T_\xi(T^*\mathrm{G}).$$

Each left-invariant Hamiltonian on $T^*\mathrm{G}$ is identified with its reduction on the dual space $\mathfrak{g}^*$.

Hamilton's equations

The equations of motion for the left-invariant Hamiltonian H are

$$\dot{g} = g\, dH(p) \quad \text{and} \quad \dot{p} = \mathrm{ad}^*_{dH(p)} p.$$

Optimal control problems

The minus Lie-Poisson structure

The dual space $\mathfrak{g}^*$ has a natural Poisson structure

$$\{F, G\}_{-}(p) = -p\left([dF(p), dG(p)]\right), \quad p \in \mathfrak{g}^*, F, G \in C^\infty(\mathfrak{g}^*).$$

If $(E_k)_{1 \le k \le m}$ is a basis for $\mathfrak{g}$ and

$$[E_i, E_j] = \sum_{k=1}^{m} c_{ij}^k \, E_k$$

then

$$\{F, G\}_{-}(p) = -\sum_{i,j,k=1}^{m} c_{ij}^k \, p_k \, \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial p_j}.$$

The Maximum Principle

The Pontryagin Maximum Principle is a necessary condition for optimality expressed most naturally in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G.

To a LiCP (with fixed terminal time) we associate - for each $\lambda \in \mathbb{R}$ and each control parameter $u \in \mathbb{R}^\ell$ - a Hamiltonian function on $T^*\mathsf{G}$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\, L(u) + \xi\left(g\Xi(\mathbf{1}, u)\right) \\ &= \lambda\, L(u) + p\left(\Xi(\mathbf{1}, u)\right), \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$.

The Maximum Principle

Theorem (Pontryagin's Maximum Principle)

Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ *is a solution for the LiCP. Then, there exists a curve* $\xi(\cdot)$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$ *and* $\lambda \leq 0$ *such that*

$$(\lambda, \xi(t)) \not\equiv (0,0) \quad (\textit{nontriviality})$$

$$\dot{\xi}(t) = \vec{H}^{\lambda}_{\bar{u}(t)}(\xi(t)) \quad (\textit{Hamiltonian system})$$

$$H^{\lambda}_{\bar{u}(t)}(\xi(t)) = \max_u H^{\lambda}_u(\xi(t)) = constant. \quad (\textit{maximization})$$

An extremal curve is called normal if $\lambda = -1$ (and abnormal if $\lambda = 0$).

Completely integrable systems

First integral

A function K on $T^*\mathsf{G}$ (or any symplectic manifold) is a first integral of a Hamiltonian system with Hamiltonian H if (and only if) $\{K, H\} = 0$.

A Hamiltonian system on $T^*\mathsf{G}$ is said to be completely integrable if there exist m $(= \dim \mathsf{G})$ first integrals $K_1, \ldots, K_{m-1}, K_m = H$ which are functionally independent (almost everywhere) and such that $\{K_i, K_j\} = 0$.

Fact

A completely integrable system can be integrated by "quadratures".
("Quadrature" means "integration of known functions".)

Completely integrable systems

For left-invariant Hamiltonian systems, there are always extra first integrals that are in involution (i.e., they Poisson commute) :

- the Hamiltonians of right-invariant vector fields
- the Casimir functions.

(NB : On semisimple matrix Lie groups, Casimir functions always exist.)

Fact

All left-invariant Hamiltonian dynamical systems on 3D (matrix) Lie groups are completely integrable.

The Lax representation

The semisimple case

If G is semisimple, then (and only then) the Killing form (on $\mathfrak{g}$) $\mathcal{K}(A, B) = \operatorname{tr}(\operatorname{ad}_A \circ \operatorname{ad}_B)$ is nondegenerate. $\mathcal{K}$ sets up a correspondence between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$: $p(\cdot) = \mathcal{K}(P, \cdot)$.

The use of the Killing form puts the eq. of motion $\dot{p} = \operatorname{ad}^*_{dH(p)} p$ in the Lax-pair form :

$$\dot{P} = [P, dH(p)], \quad P \in \mathfrak{g}.$$

Fact

The spectral invariants of P *(i.e.,* $\operatorname{tr}(P), \operatorname{tr}(P^2), \ldots, \det(P)$*) are first integrals of the (reduced) Hamiltonian system with Hamiltonian* H*.*

Optimal control problem with quadratic cost

Theorem (Krishnaprasad, 1993)

For the LiCP (with quadratic cost)

$$\dot{g} = g\ (A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$
$$g(0) = g_0,\ g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$
$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + \cdots + c_\ell u_\ell^2(t)\right)\, dt \to \min \quad (T \textit{ is fixed})$$

every normal extremal is given by

$$\bar{u}_i(t) = \frac{1}{c_i} p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the vector field* $\vec{H}$ *corresponding to* $H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1}p(B_1)^2 + \cdots + \frac{1}{c_\ell}p(B_\ell)^2\right)$.

The rotation group SO (3)

The rotation group

$$\mathrm{SO}(3) = \left\{ a \in \mathrm{GL}(3, \mathbb{R}) : a^\top a = \mathbf{1},\ \det a = 1 \right\}$$

is a 3D compact connected matrix Lie group with associated Lie algebra

$$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{so}(3)$

The standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The linear map $\widehat{\cdot} : \mathfrak{so}(3) \to \mathbb{R}^3$ defined by

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mapsto \widehat{A} = (a_1, a_2, a_3)$$

is a Lie algebra isomorphism.

We identify $\mathfrak{so}(3)$ with (the cross-product Lie algebra) $\mathbb{R}^3_\wedge$.

A drift-free left-invariant control problem

A left-invariant control problem on SO (3)

We consider the LiCP

$$\dot{g} = g\,(u_1 E_1 + u_2 E_2)\,, \quad g \in \mathsf{SO}\,(3),\; u = (u_1, u_2) \in \mathbb{R}^2$$
$$g(0) = g_0, \quad g(T) = g_1 \;\; (g_0, g_1 \in \mathsf{SO}\,(3))$$
$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t)\right)\, dt \to \min.$$

This problem appears in the modelling of spacecraft dynamics.

The dual space $\mathfrak{so}(3)^*$

$\mathfrak{so}(3)^*$ is identified with $\mathfrak{so}(3)$ via $\langle A, B\rangle = -\frac{1}{2}\,\mathrm{tr}\,(AB) = \widehat{A} \bullet \widehat{B}$.

Each extremal curve $p(\cdot)$ is identified with a curve $P(\cdot)$ in $\mathfrak{so}(3)$ via

$$\langle P(t), A\rangle = p(t)(A), \quad A \in \mathfrak{so}(3).$$

Thus

$$P(t) = \begin{bmatrix} 0 & -P_3(t) & P_2(t) \\ P_3(t) & 0 & -P_1(t) \\ -P_2(t) & P_1(t) & 0 \end{bmatrix}$$

where

$$P_i(t) = \langle P(t), E_i\rangle = p(t)(E_i), \quad i = 1, 2, 3.$$

The Lie-Poisson bracket

The (minus) Lie-Poisson bracket on $\mathfrak{so}(3)^*$ is given by

$$\begin{aligned}\{F, G\}_-(p) &= -\sum_{i,j,k=1}^{3} c_{ij}^k p_k \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial p_j} \\ &= -\widehat{P} \bullet (\nabla F \times \nabla G)\end{aligned}$$

($p \in \mathfrak{so}(3)^*$ is identified with the vector $\widehat{P} = (P_1, P_2, P_3) \in \mathbb{R}^3$).

The equations of motion

The equation of motion becomes

$$\begin{aligned} \dot{F} &= \{F, H\}_- \\ &= \nabla F \bullet \left(\widehat{P} \times \nabla H \right). \end{aligned}$$

The scalar equations of motion

$$\begin{aligned} \dot{P}_1 &= \frac{\partial H}{\partial p_3} P_2 - \frac{\partial H}{\partial p_2} P_3 \\ \dot{P}_2 &= \frac{\partial H}{\partial p_1} P_3 - \frac{\partial H}{\partial p_3} P_1 \\ \dot{P}_3 &= \frac{\partial H}{\partial p_2} P_1 - \frac{\partial H}{\partial p_1} P_2. \end{aligned}$$

The Lax-form representation

The reduced system has a Lax-form representation

$$\dot{P} = [P, \Omega]$$

where

$$P = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & 0 & \frac{1}{c_2} P_2 \\ 0 & 0 & -\frac{1}{c_1} P_1 \\ -\frac{1}{c_2} P_2 & \frac{1}{c_1} P_1 & 0 \end{bmatrix}.$$

$$C = P_1^2 + P_2^2 + P_3^2 = -\frac{1}{2} \mathrm{tr}\, \left(P^2\right)$$

is a Casimir function.

Extremal curves in $\mathfrak{so}(3)^*$

Proposition

Given the LiCS, the extremal control is $\bar{u}_1 = \frac{1}{c_1} P_1$ *and* $\bar{u}_2 = \frac{1}{c_2} P_2$*, where* $P_1, P_2 : [0, T] \to \mathbb{R}$ *(together with* P_3*) is a solution of the system*

$$\begin{aligned} \dot{P}_1 &= -\frac{1}{c_2} P_2 P_3 \\ \dot{P}_2 &= \frac{1}{c_1} P_1 P_3 \\ \dot{P}_3 &= \left(\frac{1}{c_2} - \frac{1}{c_1} \right) P_1 P_2. \end{aligned}$$

The extremal trajectories are the intersections of

- the circular cylinders $\frac{1}{c_1} P_1^2 + \frac{1}{c_2} P_3^2 = 2H$
- the spheres $P_1^2 + P_2^2 + P_3^2 = C$.

Jacobi elliptic functions

The Jacobi elliptic functions $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$, $\mathrm{dn}(\cdot, k)$ can be defined as

$$\begin{aligned}\mathrm{sn}(x, k) &= \sin \mathrm{am}(x, k)\\ \mathrm{cn}(x, k) &= \cos \mathrm{am}(x, k)\\ \mathrm{dn}(x, k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x, k)}.\end{aligned}$$

($\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is the amplitude and $F(\varphi, k) = \int_0^\varphi \frac{dt}{\sqrt{1-k^2 \sin^2 t}}$.)

Nine other elliptic functions are defined by taking reciprocals and quotients. In particular, we get

$$\mathrm{nd}(\cdot, k) = \frac{1}{\mathrm{dn}(\cdot, k)}.$$

Elliptic integrals

An elliptic integral is an integral of the type $\int R(x,y)\,dx$, where y^2 is a cubic or quartic polynomial in x and $R(\cdot,\cdot)$ denotes a rational function.

Simple elliptic integrals can be expressed in terms of inverses of appropriate Jacobi elliptic functions. Specifically (for $b \leq x \leq a$) :

$$\int_b^x \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{nd}^{-1}\left(\frac{x}{b}, \frac{\sqrt{a^2-b^2}}{a}\right)$$

$$\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{x}{a}, \frac{\sqrt{a^2-b^2}}{a}\right).$$

Explicit integration

Proposition

The reduced Hamilton equations

$$\dot{P}_1 = -\frac{1}{c_2} P_2 P_3, \quad \dot{P}_2 = \frac{1}{c_1} P_1 P_3, \quad \dot{P}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1}\right) P_1 P_2$$

can be explicitly integrated by Jacobi elliptic functions :

$$\begin{aligned} P_1 &= \pm\sqrt{\frac{c_1}{c_1 - c_2}(C - 2c_2 H - P_3^2)} \\ P_2 &= \pm\sqrt{\frac{c_2}{c_2 - c_1}(C - 2c_1 H - P_3^2)} \end{aligned}$$

Explicit integration (continuation)

and if $0 < (c_1 - c_2)P_2^2 < c_2 P_3^2$, *then*

$$P_3 = \sqrt{C - 2c_1 H} \cdot \mathrm{nd}\left(\sqrt{\frac{C - 2c_2 H}{c_1 c_2}}\, t,\ \frac{\sqrt{2(c_1 - c_2)H}}{\sqrt{C - 2c_2 H}}\right)$$

or

$$P_3 = \sqrt{C - 2c_2 H} \cdot \mathrm{dn}\left(\sqrt{\frac{C - 2c_2 H}{c_1 c_2}}\, t,\ \frac{\sqrt{2(c_1 - c_2)H}}{\sqrt{C - 2c_2 H}}\right).$$

- Similar formulas (if $c_2 P_3^2 < (c_1 - c_2)P_2^2$, etc.) can be derived.
- When $c_1 = c_2$, only circular functions are required.

Final remark

Invariant optimal control problems on matrix Lie groups other than the rotation group $SO(3)$ (like

- the Euclidean groups $SE(2)$ and $SE(3)$
- the Lorentz groups $SO_0(1,2)$ and $SO_0(1,3)$
- the Heisenberg groups $H(1)$ and $H(2)$)

can also be considered.

It is to be expected that explicit integration (of the reduced Hamilton equations) will be possible in all these cases.